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On integrals and summable trigonometric series

Suppose that the trigonometric series

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is (C,k) summable to a finite function f(x) everywhere. Subject to certain reasonable conditions on the coefficients a_n and b_n, James [8] (cf. also [3], [11]) proved that the P^{k+2}-integral [7] can be used to capture the coefficients a_n and b_n in terms of f. For example, he gave the formula for a_n as follows:

$$(*) \qquad a_n = \frac{\gamma_k}{2^{k+1}\pi^{k+2}} \int_{(\alpha_i)} f(x) \cos nx \, d_{k+2}\, x,$$

where

$$\gamma_k = \begin{cases} (2m)!/(m!)^2 & \text{if } k = 2m-2, \\ (2m+1)!/m!(m+1)! & \text{if } k = 2m-1, \end{cases}$$

and (α_i) is also chosen in a suitable way according to that k is even or odd. Here, the integral involved is the P^{k+2}-integral, which captures the "$(k+2)^{th}$ order" primitive of the integrand instead of the "first order" one, and hence the formula (*) for a_n appears to be much more complicated than the usual Euler-Fourier form. The purpose of the paper [9] is to show that for each non-negative integer k, a generalized integral, denoted as G_{k+1}-integral, of first order can be suitably defined to replace the P^{k+2}-integral in James' result so that the formula (*) will be reduced to the usual Euler-Fourier form.

To indicate what is involved for the G_{k+1}-integral, let us consider James' result for the case k = 0. James did the case in an earlier paper [6], where he showed that the P^2-integral [5], similar to the "totalisation symétrique à doux degrés" by Denjoy [4] and the T-integral by Marcinkiewicz and Zygmund [10] is powerful enough to give back the coefficients of any everywhere convergent trigonometric series in terms of its sum function. The same is true for the SCP-integral due to Burkill in [2] (cf. also [1]). The "symmetric totalisation of order two" and the P^2-integral are of second order involving essentially the concept of recapturing a function from its second symmetric derivative. The T-integral and the SCP-integral are of first order involving essentially the concept of recapturing an "integrable" function from its first "averaged" derivative; and observing the meaning of the "average" derivative, one sees that in fact both the T-integral and the SCP-integral are involving essentially the idea on how to recapture the first derivative of a function from its second symmetric derivative. Extending this idea to higher orders and based on James' work [7] and [8], the G_{k+1}-integral [9] mentioned previously is easily obtained. In fact, it is involving essentially the concept of recapturing the $(k+1)^{\underline{th}}$ Peano derivative of a function from its $(k+2)^{\underline{th}}$ generalized symmetric derivative, and is defined by the descriptive method of Perron. For details, we refer to [9].

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