*Real Analysis Exchange Vol. 4 (1978-79)*
Togo Nishiura, Department of Mathematics, Wayne State University, Detroit, Michigan 48202.

## Some Examples Relating Surface Area and the Area Integral.

A well developed theory of area for discontinuous nonparametric surfaces has been discussed by C. Goffman in the main lectures of the Symposium on Real Analysis. The theory of discontinuous parametric surfaces is not as well developed. As pointed out by Goffman, the very definition of area is not at all obvious because convergence in measure is not enough to make area a lower semicontinuous functional in the class of piecewise smooth maps. In [1], Goffman and Liu show that linear continuity is not the right concept for parametric surface area even thought it does have a natural place in the nonparametric case. They further show that, for mappings from $R^n$ into $R^m$ $(m \geq n)$, the notion of $(n-1)$-continuity does work. We discuss briefly this notion next.

For convenience we will assume that our mappings are defined on an n-cube in $R^n$. A map $f:X \to R^m$ is said to be <u>$(n-1)$-continuous</u> if its restriction to almost every $(n-1)$-hyperplane orthogonal to a coordinate direction is continuous. Clearly, continuous maps are $(n-1)$-continuous. There is a natural metric on this class of maps called the $(n-1)$-continuity metric. (It is really a pseudo metric). It corresponds to uniform convergence on almost every $(n-1)$-hyperplane involved in the definition of this class of maps. The class $\mathcal{P}$ of piecewise smooth maps is a dense subspace and its closure is precisely the class of (n-1)-continuous maps. As mentioned above, area is a lower semicontinuous functional on $\mathcal{P}$ with respect to this metric and hence has a Fréchet extension to the class of (n-1)-continuous maps. Denote by $A(f)$ this area functional. It is proved in [1] that $A(f)$ is equal to Lebesgue area whenever $f$ is continuous.

The paper [2] by Goffman and Ziemer has influenced the direction of research for (n-1)-continuous maps. It established that a certain type of Sobolev map is (n-1)-continuous and that the area of such a map is given by the classical area integral. Further research was done along this line by Goffman and Liu [3]. In both works, mollification leads to smooth maps which converge in the (n-1)-continuity metric to the Sobolev map and whose Jacobians converge in the $L_1$-norm to the formal Jacobian of the Sobolev map. Not all Sobolev maps are of the type investigated in [2] and [3] as shown by the example of D. Pepe [4]. Motivated by the above papers, Breckenridge and the author [5] investigated the relationship of area and the integral formula for the area by studying another metric on the class $\mathcal{P}$ of piecewise smooth maps. We describe this metric below.

Denote by $d(f,g)$ the (n-1)-continuity metric. Let $f_1$ and $f_2$ be piecewise smooth maps from $X$ into $R^m$ and $Jf_1$ and $Jf_2$ be the Jacobians of these maps. We define the metric $\rho(f_1,f_2)$ on $\mathcal{P}$ by

$$\rho\ (f_1,f_2) = d(f_1,f_2) + \int_X |Jf_1 - Jf_2|\ d\,\mu_n\ .$$

The completion of $\mathcal{P}$ with respect to this metric will be denoted by $D^*[n,m]$. Our interest is in the class of continuous maps $c[n,m]$ . Since mollification in this case leads to uniform convergence, we will also use the uniform norm as follows.

$$\rho_\infty (f_1,f_2) = \|f_1 - f_2\|_\infty + \int_X |Jf_1 - Jf_2| d\,\mu_n .$$

The completion of $\mathcal{P}$ with respect to $\rho_\infty$ will be denoted by $cD^*[n,m]$ . This space can be identified as a subset of $c[n,m] \times L_1(X, \wedge^n R^m)$ . $(f,J_*f)$ will denote members of $cD^*[n,m]$. If $f$ has a formal Jacobian $Jf$ almost everywhere and $(f,Jf) \in cD^*[n,m]$ , we say $f \in cD[n,m]$ . So continuous maps of the type of Goffman, Ziemer and Liu are in $cD[n,m]$ . Also, any continuous map of zero area is in $cD^*[n,m]$ , where $J_*f = 0$ .

Our paper [5] left unresolved the problem of the uniqueness of $J_*f$ . That is, are there two functions $F_1$ and $F_2$ corresponding to $f$ so that $(f,F_1)$ and $(f,F_2)$ are both in $cD^*[n,m]$? Sufficient conditions for uniqueness can be found in [5]. We give three examples, all very different from each other.

<u>Example</u> 1. Let $X$ be the unit square in $R^2$ and $f:X \to R^2$ be the constant map $f(x) = 0$ . Clearly, $f \in cD[2,2]$ and $Jf = 0$ . It can be shown that for any summable function $F:X \to R^1$ we have $(f,F) \in cD^*[2,2]$ . We illustrate the construction for $F = 1$ . Consider the four subsets of $R^2$ in Figure 1 below. We have $\operatorname{diam} W < \frac{1}{2}$, $Z \subset Y \subset X$ and a piecewise linear retraction $\varphi$ of $X$ onto $Y$ . There is a piecewise linear map $\psi$ of $Y$ onto $Z$ which maps the squares of each row of $Y$ affinely onto the squares

of $Z$. Finally there is a piecewise linear map $\xi$ of $Z$ onto $W$ which maps each square of $Z$ affinely onto $W$. Let $f_1 = \xi \circ \psi \circ \varphi$. Then $\|f - f_1\|_\infty < \frac{1}{2}$ and $Jf_1(x) = 1$ for almost every $x$ in $Y$ and $Jf(x) = 0$ for almost every $x$ in $X - Y$. Hence

$$\int_X |Jf_1 - 1| d\,\mu_2 = \mu_2(X - Y) ,$$

which can be made arbitrarily small. One can see that there is a sequence $f_k \in \mathcal{P}$ such that $\|f - f_k\|_\infty \to 0$ and

$$\int_X |Jf_k - 1| d\,\mu_2 \to 0$$

as $k \to \infty$. That is, $(f,1) \in cD^*[2,2]$. Note that for any scalar $t$, we have $(f,t^2) \in cD^*[2,2]$ because $J(tf_k) = t^2 Jf_k$. It is not difficult to show $(f,-t^2) \in cD^*[2,2]$.

Let $g$ be the orthogonal projection of $X$ onto $\{0\} \times [0,1]$ and $G$ be any summable function on $X$. Then one can easily see that $(g,G) \in cD^*[2,2]$.

Each of the two continuous maps $f$ and $g$ above are in $\mathcal{P}$ and have area zero. Also, $Jf$ and $Jg$ exist almost everywhere and are equal to zero. One observes that for any $J_*f$ for which $A(f) = \int_X |J_*f| d\,\mu_2$ we have $J_*f = Jf$ almost everywhere. This is because $A(f) = 0$. The same is true for $g$ above.

<u>Example</u> 2. This example deals with maps with positive area. Let $X$ and $X_o$ be concentric squares in $R^2$ and $f:X \to R^2$ be the piecewise linear map which collapses the inner square $X_o$ to the center and maps $X$ onto $X$. This map $f$ is in $cD[2,2]$,

$Jf(x) = 0$ for $x \in X_o$ and $A(f) = \int_X |Jf| d\mu_2 = 1$. Using the example 1 above, one can easily construct a $J_*f$ such that $J_*f(x) = Jf(x)$ for $x \in X - X_o$ and $J_*f(x) \neq 0$ for almost every $x \in X_o$. Hence $(f,Jf) \neq (f,J_*f)$. We know [5;Theorem 3.1] that $A(f) \leq \int_X |J_*f| d\mu_2$ and [5;Theorem 5.1] that $|Jf| \leq |J_*f|$ almost everywhere for any choice of $J_*f$ with $(f,J_*f) \in cD^*[2,2]$. Hence, if $A(f) = \int_X |J_*f| d\mu_2$ then $J_*f = Jf$ almost everywhere. Another interesting fact about this $f$ is that $\int_{X_o} J_*f d\mu_2 = 0$ whenever $(f,J_*f) \in cD^*[2,2]$. (See [5; Theorem 3.3].)

<u>Example</u> 3. There is an example $f:X \to R^2$, where $X$ is a rectangle in $R^2$ and two summable functions $F_1$, $F_2$ with $\int_X |F_1 - F_2| d\mu_2 > 0$, $(f,F_1) \in cD^*[2,2]$, $(f,F_2) \in cD^*[2,2]$ and $A(f) = \int_X |F_1| d\mu_2 = \int_X |F_2| d\mu_2$.

Thus, the condition $A(f) = \int_X |J_*f| d\mu_n$ does not determine $J_*f$ uniquely in $cD^*[n,n]$.

Consider a map $f:X \to R^2$ described as follows. Let $C$ be an arc in $I^2$ with positive measure which separates $I^2$ into components $A$ and $B$ as in Figure 2 below. Let $X$ be a rectangle which is divided into three components by arcs $C_1$ and $C_2$, the three components being labeled $A_1$, $B_1$ and $D$. The restrictions $f|(A_1 \cup C_1)$ and $f|(B_1 \cup C_2)$ are isometries onto $A \cup C$ and $B \cup C$ respectively. $f^{-1}(y)$ is an arc in $C_1 \cup D \cup C_2$ for each $y \in C$. Such maps $f$ are used in [6] by L. Cesari for another purpose. One can construct such a map so that if $F_1$ is the characteristic function of $A_1 \cup C_1 \cup B_1$ and $F_2$ is the characteristic function of $A_1 \cup C_2 \cup B_1$ then $(f,F_1) \in cD^*[2,2]$ and $(f,F_2) \in cD^*[2,2]$. Since $C_1$ and $C_2$ are isometric with $C$ we have $\int_X |F_1 - F_2| d\mu_2 = 2\mu_2(C) > 0$. Also, $\int_X |F_1| d\mu_2 = 1$, $\int_X |F_2| d\mu_2 = 1$ and $A(f) = 1$. (The above example was suggested to the author by J.C. Breckenridge.) We remark that the Jacobian $Jf$ for $f$ does not exist almost everywhere.

Addendum. The following example is given to illustrate that "pathological" (n-1)-continuous surfaces can exist.

Let $X = [0,1] \times [0,1]$ and $g:\partial X \to R^m$ be any continuous map from the boundary of $X$. Let $r:X-\{p\} \to \partial X$ be the natural retraction of $X-\{p\}$ onto $\partial X$ where $p$ is the center of the square $X$. Then $f:X \to R^m$ given by $f(x) = g \circ r(x)$ for $x \in X-\{p\}$ and $f(p) = g(0)$ is (n-1)-continuous for $n=2$. Notice that $f(X) = g(\partial X)$ is not a two-dimensional object. If $g$ is one-to-one, then $A(f) > 0$. Moreover, let $F:X \to R^m$ be any continuous map for which $F|\partial X = g$. Then we assert $A(f) \le A(F)$. To see this, let $X_k$ be a square concentric with $X$ with diameter less than $1/k$ and let $\eta_k:X \to X$ be the continuous map which retracts $X-X_k$ onto $\partial X$ and maps $X_k$ affinely onto $X$. Consider $F_k = F \circ \eta_k$. One gets $A(F_k) = A(F)$ and $d(F_k,f) \to 0$ as $k \to \infty$. Hence by the lower semicontinuity of the area $A$ with respect to the metric d, we have

$$A(f) \le \liminf_{k \to \infty} A(F_k) = A(F).$$

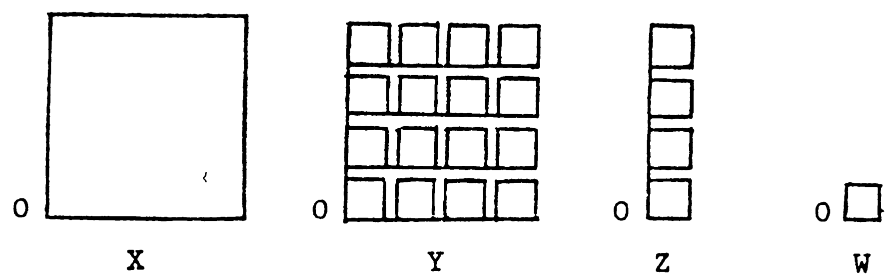
Thus, we find that

$$A(f) \le \inf\{A(F) \mid F:X \to R^m \text{ is continuous}, \ F|\partial X = g\}.$$

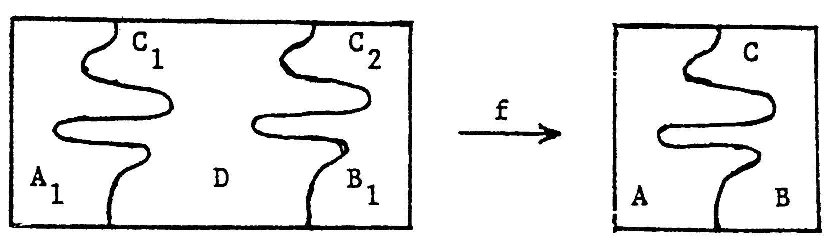
That is, there is a connection with the minimum area problem.

<u>Conjecture</u>: The above inequality is actually an equality.

<u>Figure</u> 1.




<u>Figure</u> 2.



References

1. C. Goffman and F-C. Liu, Discontinuous mappings and surface area, Proc. London Math. Soc. 20 (1970), 237-248.

2. C. Goffman and W.P. Ziemer, Higher dimensional mappings for which the area formula holds, Ann. of Math. 92 (1970), 482-488.

3. C. Goffman and F-C. Liu, The area formula for Sobolev mappings, Indiana U. Math. J. 25 (1976), 871-876.

4. D. Pepe, Mollification does not imply convergence in area, Indiana U. Math. J. 23 (1973/74), 273-276.

5. T. Nishiura and J.C. Breckenridge, Differentiation, integration, and Lebesgue area, Indiana U. Math. J. 26 (1977), 515-536.

6. L. Cesari, Surface Area, Ann. of Math. Studies, no. 35, Princeton Univ. Press, Princeton, N.J., 1956.

*Received October 18, 1978*