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On a Class of Orthogonal Series

In [2], Skvorcov introduced a generalization of the Perron integral for the purpose of calculation of the coefficients of a Haar series. I would like to mention some results of J. C. Georgiou and myself which extend Skvorcov's theorems to a wider class of orthogonal series. Some related questions have been studied, e.g., in [4] and [5].

1. Let V be a real vector space and let S be a subspace of V. Suppose that φ is a function on $S \times V$ such that $\varphi(s,.)$ is linear on V for each $s \in S$, $\varphi(.,v)$ is linear on S for each $v \in V$, $\varphi(s,s) > 0$ for each $s \in S\setminus\{0\}$ and that $\varphi(s,v) = \varphi(v,s)$, whenever $s, v \in S$. The restriction of φ to $S \times S$ is, obviously, an inner product so that we may speak about orthogonality in S.

Let T be a finite-dimensional subspace of S and let $v \in V$. It is easy to see that there is a unique $p \in T$ such that $\varphi(t,v) = \varphi(t,p)$ for each $t \in T$; write $p = \text{o.p.}(v,T)$ (orthogonal projection of v to T). If $T_0, T_1, \ldots$ are pairwise orthogonal finite-dimensional subspaces of S and if $v \in V$, then $\sum_{n=0}^{\infty} \text{o.p.}(v,T_n)$ will be called the Fourier series of v with respect to the sequence $\langle T_n \rangle$.

2. Let $D_0, D_1, \ldots$ be finite subsets of $[0,1]$ such that $\{0,1\} \subset D_0 \subset D_1 \subset \ldots$ and that $D_0 \cup D_1 \cup \ldots$ is dense in $[0,1]$. If we partition $[0,1]$ by D_n, we get a system of closed intervals which will be denoted by $\mathcal{D}_n$. Let S_n be the system of all functions f on $[0,1]$ such that f is constant on $\operatorname{int} J$ for each $J \in \mathcal{D}_n$, $f(0+) = f(0)$, $f(1-) = f(1)$ and $f(x) = \frac{1}{2}(f(x+) + f(x-))$ for each $x \in (0,1)$. Obviously $S_0 \subset S_1 \subset \ldots$. Define $S = S_0 \cup S_1 \cup \ldots$ and introduce in S an inner product in the usual way. Let $T_0 = S_0$ and let T_n be the orthogonal complement of S_{n-1} in S_n for $n = 1,2,\ldots$. For each $x \in [0,1)[x \in (0,1]]$ let $J_n(x)[J_n'(x)]$ be the element $[a,b]$ of $\mathcal{D}_n$ for which $x \in [a,b)[x \in (a,b]]$; further set $J_n(1) = \{1\}$, $J_n'(0) = \{0\} (n = 0,1,\ldots)$.

3. Let V be a vector space whose elements are functions on $[0,1]$ and let L be a linear functional on V with the following properties: If f is a finite Lebesgue integrable function on $[0,1]$, then $f \in V$ and Lf is its integral; if $s \in S$ and $v \in V$, then $sv \in V$. It is obvious that all the assumptions of 1 are fulfilled, if we take $\varphi(s,v) = L(sv)$. It is easy to prove the following assertion:

Let n be a nonnegative integer. Let $f \in V$, $J \in \mathcal{D}_n$, $x \in \operatorname{int} J$ and let c be the characteristic function of J.

Set $s_n = \sum_{k=0}^{n} \text{o.p.}(f, T_k)$. Then $s_n = \text{o.p.}(f, S_n)$ and $s_n(x) = |J|^{-1} \cdot L(fc)$ (if $J = [a,b]$, then $|J| = b - a$).

4. In [2], Skvorcov constructed an integral that integrates the sum of each everywhere convergent Haar series $\sum a_n \chi_n$ for which

$$(1) \qquad a_n / \chi_n(x) \longrightarrow 0 \quad (n \to \infty, \chi_n(x) \neq 0) .$$

It is possible to generalize Skvorcov's result in various ways. To illustrate the matter suppose that the set $D_{n+1} \cap \text{int } J$ has at most one point for each $J \in \mathcal{D}_n$ and that there is a number $q > 0$ such that $|K| > q|J|$, whenever $J \in \mathcal{D}_n$, $K \in \mathcal{D}_{n+1}$ and $K \subset J (n = 0,1,\ldots)$. Then there are V and L fulfilling the assumptions of 3 such that the following theorem holds:

Let $f_n \in T_n$, $s_n = \sum_{k=0}^{n} f_k$. Let

$$(2) \qquad \int_{J_n(x)} s_n \longrightarrow 0, \quad \int_{J_n'(x)} s_n \longrightarrow 0 \quad (n \to \infty)$$

for each $x \in [0,1]$ and let the set $\{x ; \sup_n |s_n(x)| = \infty\}$ be countable. Then there is an $f \in V$ such that $\sum_{n=0}^{\infty} f_n(x) = f(x)$ almost everywhere and that $\sum_{n=0}^{\infty} f_n$ is the Fourier series of f with respect to $\langle T_n \rangle$.

In the proof we apply methods developed in [2] and [3] and a theorem proved in [1].

5. Now suppose that D_n has exactly $n + 2$ points. Then T_n has dimension 1; let g_n generate T_n and let $\int_0^1 g_n^2 = 1 (n = 0,1,\ldots)$. We may choose $g_0 = 1$. Now let $n > 0$, $p \in D_n \setminus D_{n-1}$ and $p \in J = [a,b] \in \mathcal{D}_{n-1}$. Then we may choose g_n in such a way that $g_n > 0$ on (a,p). If $D_1 = \{0, \frac{1}{2}, 1\}$, $D_2 = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$, $D_3 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, $D_4 = \{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}, \ldots,$ then $g_n = \chi_n$ (the Haar function) for each n. It is not difficult to prove that, in this case, (1) is equivalent to (2).

6. Finally, let $D_n = \{k.2^{-n}; k = 0,1,\ldots 2^n\}$, let $\psi_0, \psi_1, \ldots$ be the Walsh functions and let f be a Perron integrable function on $[0,1]$. Let $\Sigma\, a_n \chi_n$ and $\Sigma\, b_n \psi_n$ be the Haar - and Walsh - Fourier series of f, respectively. Let n be a nonnegative integer and let $m = 2^n$. As $\chi_0, \ldots, \chi_{m-1}$ is an orthonormal basis of S_n and as the same is true for $\psi_0, \ldots, \psi_{m-1}$, we have

$$\sum_{k=0}^{m-1} a_k \chi_k = \text{o.p.}(f, S_n) = \sum_{k=0}^{m-1} b_k \psi_k \quad \text{(see [4])}.$$

References

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