INROADS *Real Analysis Exchange Vol. 4 (1978-79)*

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Iterated L^p Derivatives

If a function f has a k-th derivative in the usual sense at x, then according to the classical Taylor Theorem,

$$f(x+t) - f(x) - t f'(x) - \cdots - (t^k/k!) f^{(k)}(x) = o(t^k)$$

as t tends to 0. When such a formula holds (which is possible without $f^{(k)}(x)$ existing) f is said to have a k-th Peano derivative. More specifically f has a k-th Peano derivative at x if there are k-numbers, $f_1(x), \cdots, f_k(x)$ such that

$$f(x+t) - f(x) - t f_1(x) - \cdots - (t^k/k!) f_k(x) = o(t^k)$$

as t tends to 0. Interpret this condition as saying that the supremum of the left hand side over t between 0 and h; that is, the L^∞-norm of the left hand side on the interval between 0 and h, is $o(h^k)$ as h tends to 0. It is natural to replace L^∞-norm by L^p-norm, but it must be normalized so that the function identically 1 has L^p-norm 1. When this is done we have the definition of the k-th derivative in L^p. Precisely, f is said to have a k-th derivative in L^p, $0 < p < \infty$, at x if there are numbers $f_{p,1}(x), \cdots, f_{p,k}(x)$ such that

$$\{\frac{1}{h}\int_0^h |f(x+t)-f(x)-t\,f_{p,1}(x)-\cdots-(t^k/k!)\,f_{p,k}(x)|^p dt\}^{1/p} = o(h^k)$$

as h tends to 0. Clearly if f has a k-th Peano derivative at x, then f has a k-th derivative in L_p for any $0 < p < \infty$, and $f_{p,i}(x) = f_i(x)$ for all $i = 1,\cdots,k$. So for consistancy of notation we denote the Peano derivatives by $f_{\infty,i}(x)$; that is, $f_i(x) = f_{\infty,i}(x)$ for $i = 1,\cdots,k$. Now if f has a k-th derivative in L^p at x for some $0 < p$, then it can be shown that there is a set E having density 1 at 0 such that

$$f(x+t)-f(x)-t\,f_{p,1}(x)-\cdots-(t^k/k!)\,f_{p,k}(x) = o(t^k)$$

as $t \in E$ tends to 0. Consequently, we say that f has a k-th approximate Peano derivative at x if there are numbers $f_{0,1}(x),\cdots, f_{0,k}(x)$ and a set E having density 1 at 0 such that

$$f(x+t)-f(x)-t\,f_{0,1}(x)-\cdots-(t^k/k!)\,f_{0,k}(x) = o(t^k)$$

as $t \in E$ tends to 0. This gives us a k-th differentiation method for each $0 \le p \le \infty$ which is stronger as p increases.

As indicated by the title we deal with iterated derivatives, and determined when a j-th derivative in L^q of a k-th derivative in L^p is a $k+j$-th derivative and of what scale.

Theorem: Suppose that f has a k-th derivative, $f_{p,k}$, in L^p, $0 \leq p \leq \infty$, on a neighborhood of x and that $(f_{p,k})_{q,j}(x)$ exists $1 \leq q \leq \infty$. Then f has a $k+j$-th Peano derivative at x and

$$f_{\infty,k+j}(x) = (f_{p,k})_{q,j}(x).$$

There are two examples which demonstrate just how crucial the condition $q \geq 1$ is in the theorem. First there is a function f such that for each $q < 1$, the strongest possible iterated derivative exists, namely $(f_{\infty,1})_{q,1}(0) = 0$ but the weakest 2-nd derivative does not; that is, $f_{0,2}(0)$ does not exist. Second there is a function f such that for each $q < 1, (f_{\infty,1})_{q,1}(0)$ and $f_{\infty,2}(0)$ both exist but they are not equal. So the condition $q \geq 1$ is needed not only to insure the existence of the noniterated derivative but also to be assured of the equality of the iterated and noniterated derivatives. The examples are similar.

We define $f(x)$ for $x \geq 0$, and extend f by $f(-x) = f(x)$. Let $\{[a_n,b_n]\}$ be a sequence of pairwise disjoint closed intervals decreasing to 0 as n tends to ∞, and let $\{c_n\}$ be a sequence of positive numbers decreasing to 0 as n tends to ∞. For each n, let f be constant, c_n, between b_n and a_{n-1}. (Let $a_0 = +\infty$.) On $[a_n,b_n]$ let f increase in a differentiable fashion from c_{n+1} to c_n. Let $f(0) = 0$. For the first example let $a_n = 2^{-n}$, $b_n = a_n(1 + n^{-1})$, and $c_n = 4^{-n}$. For the second let $a_n = n^{-1}$, $b_n = a_n(1 + n^{-1})$, and $c_n = (n+1)^{-3}$.

Received October 27, 1978