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Bounded Variation and Absolute Continuity in the Theory of Surface Area

by

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This survey is a compilation of the lectures on functions useful in surface area given by Professor Goffman at the SYMPOSIUM IN REAL ANALYSIS at Western Illinois University, July 19-23, 1978. It has been prepared and edited by the editorial board of the Exchange from Professor Goffman's lecture notes and from video tapes of his lectures. For cohesiveness, the final lecture on bounded variation and absolute continuity in the theory of Fourier series has been omitted. The lectures and this survey are expository in nature, intended to provide insight into the theory. Nevertheless, several proofs and indications of proofs are included as a vehicle to further this insight. The reader in search of detailed proof for a particular result included here should consult the appropriate bibliographic reference.

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§0. Introduction

Functions of bounded variation were introduced by Jordan in the last century and were found by him to have importance both for the theory of arc length and for the convergence of Fourier series.

In the third and fourth decades of this century, Tonelli and Cesari introduced appropriate notions of absolute continuity and bounded variation to handle two dimensional surface area and double Fourier series.

Much has been done during the last decade to obtain analogous parallels in higher dimensions involving Sobolev spaces on the one hand and refined spaces of functions of bounded variation on the other. The results are mostly for the area formula and for localization for multiple Fourier series.

The main purpose of these notes is to develop needed properties of these spaces. Little attention is given to the applications themselves.

§ 1.

In this section we consider three classes of real valued functions defined on an interval $I = [a,b]$. These are the bounded variation functions, the continuous bounded variation functions, and the absolutely continuous functions. The three classes are designated by BV, CBV, and AC, respectively. Analogues of these classes for functions of several variables are important in the development of surface area theory. Hence, it seems appropriate to begin by considering some of the properties of these classes in the one variable setting. Proofs for these properties are outlined here for a couple of reasons - 1) the proofs are considerably simpler than those of the corresponding several variables results and 2) hopefully the one variable proofs will provide insight into what lies ahead.

In order to give a perspective for the analogous definitions in higher dimensions, we characterize these three classes in terms of distributions. A distribution is a continuous linear functional on the vector space of continuously differentiable functions having compact support where the vector space topology is suitably chosen. The derivative L' of a distribution L is defined by $L'(\varphi) = -L(\varphi')$ for $\varphi \in C^{\infty}$. Thus, every distribution has a derivative. This definition is motivated by the case where L is determined by a continuously differentiable function f, i.e., $L(\varphi) \equiv \int f(x)\varphi(x)dx$. Then, the integration by parts formula yields

$$L'(\varphi) = \int f'(x)\varphi(x)dx = - \int f(x)\varphi'(x)dx = -L(\varphi').$$

Now, L is given by an absolutely continuous function if and only if L' is given by a summable function; L is given by a bounded variation function if and only if L' is given by a totally finite measure; and L is given by a continuous bounded variation function if and only if L' is given by a non-atomic totally finite measure.

We shall first note that functions in AC may be characterized as those which can be approximated in a certain sense by functions in C^1. More specifically, let f be defined on an interval J and let $\ell(f,J)$ denote the arc length of the curve determined by f over J. For an open set $G = \cup I_n$, where the I_n are pairwise disjoint, we define $\ell_f(G) = \ell(f,G) = \sum_n \ell(f,I_n)$. This set function generates a length measure $\ell_f(E)$ defined on the Borel sets in I. If f and g are such that $\ell_f(I) < \infty$, $\ell_g(I) < \infty$, and $f(x) = g(x)$ for every x in some Borel set E, then by a result of Verchenko [15], $\ell_f(E) = \ell_g(E)$. In the present one dimensional case this follows from the fact that length measure equals Hausdorff one dimensional measure.

THEOREM 1. _A function_ $f\colon I\to R$ _belongs to_ AC _if and only if, for each_ $\epsilon > 0$ _there is a_ $g \in C^1$ _such that if_ $G = \{x \mid f(x)\neq g(x)\}$, _then_ $\ell_f(G) < \epsilon$ and $\ell_g(G) < \epsilon$.

Proof. To prove the sufficiency, suppose f satisfies the condition. Then $f \in BV$ and hence $\ell_f(I) < \infty$. Let $\epsilon > 0$; let g be the function of the theorem for this ϵ; and let $G = \{x \mid f(x)\neq g(x)\}$. Then

$$\ell_f(I) = \ell_f(G) + \ell_g(I\setminus G) \le \int_{I\setminus G} \sqrt{1+[f'(x)]^2}\,dx + \epsilon,$$

whence $\ell_f(I) \le \int_I \sqrt{1+[f'(x)]^2}\,dx$ and $f \in AC$.

For the converse, suppose $f \in AC$. Let $\epsilon > 0$. By a customary real variables argument, there is a perfect set $E \subset I$ whose complement has ℓ_f measure less than $\epsilon/2$, such that f is uniformly continuously differentiable on E, in the sense that for each $\eta > 0$ there is a $\delta > 0$ such that, for every interval $J \subset I$ of length less than δ and for every two pairs of points in $J \cap E$, the corresponding difference quotients of f differ by less than η. We can then extend f from E to I by defining it to be linear on the complementary intervals of E and then "rounding off" at the end points in such a way that the resulting g satisfies the desired conditions.

Theorem 1 also follows from a deep theorem of J. H. Michael [12], a form of which will be described in §2. of this article.

There is a companion result for functions in BV, and it has a similar proof.

THEOREM 2. _If_ $f: I \to R$ _belongs to_ BV, _then for each_ $\varepsilon > 0$ _there is a_ $g \in C^1$ _such that_ $f(x) = g(x)$ _except on a set of measure less than_ ε _and_ $|\ell_f(I) - \ell_g(I)| < \varepsilon$.

Proof. As before, there is a perfect $E \subset I$ such that $m(I \setminus E) < \varepsilon$ and f is uniformly continuously differentiable on E. We extend f as before to a continuously differentiable function h on I in such a way that $\ell_h(I) < \ell_f(I) + \varepsilon$. Pick one open interval J of $I \setminus E$ and modify h on J to a function $g \in C^1$ such that $|\ell_f(I) - \ell_g(I)| < \varepsilon$.

The third one variable result we discuss has the most interest for us. It is a theorem of F. C. Liu [11]. Consider the compactification $\overline{R}$ of R obtained by adjoining $-\infty$ and $+\infty$. A function $f: I \to R$ is called _weakly continuously differentiable_ if the derivative exists or is $+\infty$ or $-\infty$ at each $x \in I$ and is a continuous function from I into $\overline{R}$.

THEOREM 3. *If* $f \in CBV$ *then for each* $\epsilon > 0$ *there is a weakly continuously differentiable function* g *such that if* $G = \{x | f(x) \neq g(x)\}$, *then* $\ell_f(G) < \epsilon$ *and* $\ell_g(G) < \epsilon$.

We shall not give the proof but shall make some instructive remarks. Accordingly, consider the Cantor function f. Now, $I = [0,1]$ and $f(I) = [0,1]$. Let $E_1 \subset I$ and $E_2 \subset f(I)$ be perfect sets such that $f(E_1) \cap E_2 = \emptyset$, $m(E_1) > 1 - \epsilon/2$, $m(E_2) > 1 - \epsilon/2$, f is uniformly continuously differentiable on E_1, and f^{-1} is uniformly continuously differentiable on E_2. We may then extend f from $E_1 \cup f^{-1}(E_2)$ to a weakly continuously differentiable g as desired.

Suppose that by a similar argument we have the theorem for monotone functions. It is natural to use the decomposition of f as a difference $f = f_1 - f_2$ of increasing functions, obtain g_1 and g_2 as above, and let $g = g_1 - g_2$. The danger is that these functions could have infinite derivatives at the same point. It seems plausible that a satisfactory pair f_1, f_2 could be obtained by using a Hahn decomposition. Let E and F be disjoint Borel sets in I such that the variation measure is non-negative on every subset of E, non-positive on every subset of F, and zero on every subset of $I-(E \cup F)$. Let $f_1(x) = m[(a,x) \cap E]$ and $f_2 = f_1 - f$. It should then be possible to define g_1 and g_2 as needed.

The actual proof of Theorem 3 as given by Liu is quite delicate.

§ 2.

This section is devoted to a brief study of generalizations of the notion of bounded variation to higher dimensions. In particular, we will discuss bounded variation in the sense of Cesari, giving appropriate equivalent definitions, and the specific properties of functions in these classes which will be important to subsequent discussions. Finally, associated work of Federer and Goffman is examined.

Let Q denote the unit n cube. A function f is of bounded variation in the sense of Cesari ($f \in BVC$) if f is summable and if for every $i = 1, 2, \ldots, n$ there is a function g^i which is equivalent to f and which is of bounded variation in x_i for almost all values of the other n-1 variables and the variation function is a summable function on the n-1 cube. In a, b, and c, to follow, we state several properties and equivalences.

a. A function f is in BVC if and only if the partial derivatives, (in the distribution sense) of f are totally finite measures. That is, there is a vector valued measure $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ such that for every $\varphi \in C^\infty$ we have

$$\int \varphi(x)\, d\mu_i(x) = - \int \frac{\partial \varphi(x)}{\partial x_i} f(x)\, dx, \quad i = 1, 2, \ldots, n.$$

This was first noticed by Krickeberg [10] and later a simpler proof was given by Goffman and Serrin [9].

b. If $f \in$ BVC, there is a function g (independent of i) which is equivalent to f and which is of bounded variation in $x_i (i=1,2,\ldots,n)$ for almost all values of the other n-1 variables, and the variation functions are all summable functions of the n-1 cube. Indeed, it can be shown that g has this property for <u>all</u> directions; this fact was first proved by Serrin and Hughs independently. Each proof depends upon the fact that for any given coordinate direction, the properly chosen equivalent function is the difference between two summable functions, each of which is nondecreasing as a function of one variable in this direction for almost all values of the other variables. It is interesting that these functions, although summable, need not be in BVC. The required function is actually the lim sup of functions which have been regularized via circularly symmetric regularizers.

c. A theory of surface area was given by Cesari [1] for such functions and later Goffman [4] extended these ideas and developed many properties of these functions. We shall give some indication of this work. A real function p on Q is piecewise linear if it is continuous and if there is a simplicial decomposition of Q such that on each simplex σ in this decomposition, p is linear. The area $a(p)$ is defined by

$$a(p) = \Sigma v(p(\sigma))$$

where $v(p(\sigma))$ is the n volume of the simplex $p(\sigma)$. This particular area functional was extended by Lebesgue to all continuous functions f on Q. This extension is accomplished by noting that $a(p)$ is lower semi-continuous on the metric space of piecewise linear functions (denoted by P) where the distance is defined as

$$d(p,q) = \max_{x \in Q} |p(x)-q(x)|.$$

Every lower semi-continuous functional on a metric space has an extension to a maximal lower semi-continuous functional on its completion. The completion of P is the space C of continuous functions on Q, and the corresponding extended functional, $A(f)$, is the Lebesgue area.

This completes our introduction to Cesari's theory and we now consider Tonelli's theory and its generalizations. Tonelli [14] not only introduced a generalized notion of bounded variation, but also introduced a suitable generalization of absolute continuity and both of these notions play a major role in the theory. A function f is of bounded variation in the sense of Tonelli ($f \in BVT$) if $f \in BVC$ and f is continuous. The function f is absolutely continuous in the Tonelli sense ($f \in ACT$) if $f \in BVT$ and if f is absolutely continuous in each variable for almost all values of the other variables. If $f \in BVT$ then its partial derivatives exist almost everywhere. Tonelli's main theorems are listed below as Theorem T_1 and Theorem T_2.

THEOREM T_1. _The area functional_ $A(f)$ _is finite if and only if_ $f \in BVT$, _and then_

$$A(f) \geq \int_Q [1 + f_{x_1}^2 + f_{x_2}^2 + \ldots + f_{x_n}^2]^{1/2} dx_1 \ldots dx_n.$$

THEOREM T_2. _The function_ $f \in ACT$ _if and only if_

$$A(f) = \int_Q [1 + f_{x_1}^2 + f_{x_2}^2 + \ldots + f_{x_n}^2]^{1/2} dx_1 \ldots dx_n.$$

The method of extension employed by Goffman [4] is to alter the metric on P in a suitable way. First consider the coarser distance given by

$$\delta(p,q) = \int_Q |p(x) - q(x)|dx.$$

The functional $a(p)$ is again lower semi-continuous on P. (This is easy to show using the fact that convergence in L_1 implies convergence in measure.) Now, consider the extension of $a(p)$ to a lower semi-continuous functional $E(f)$ defined on the space of equivalence classes of summable functions. An important and interesting relationship is that $A(f) = E(f)$ if f is continuous, and so the new area functional is an extension of Lebesgue area. We indicate the proof of this. Clearly $E(f) \leq A(f)$. The opposite inequality follows from the fact that if f is continuous and $A(f) < \infty$, then for every $\epsilon > 0$ there is an $\eta > 0$ such that if g is continuous and $|f(x) - g(x)| < \eta$ except on a set of measure less than η, then $A(g) > A(f) - \epsilon$. A similar fact holds if $A(f) = \infty$.

We now consider the area functional E in the extended theory. The role BVT played for Tonelli is now played by BVC, and the role of ACT is now played by those functions whose partial derivatives, in the distribution sense, are summable functions. These functions are the Sobolev spaces W_1^1 and we now have the following theorems.

THEOREM G_1. <u>The area functional</u> $E(f)$ <u>is finite if and only if</u> $f \in$ BVC, <u>and then</u>

$$E(f) \geq \int_Q [1 + f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2]^{1/2} \, dx_1 \dots dx_n.$$

THEOREM G_2. <u>The function</u> $f \in W_1^1$ <u>if and only if</u>

$$E(f) = \int_Q [1 + f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2]^{1/2} \, dx_1 \dots dx_n.$$

d. A function f may be in W_1^1 and yet be everywhere discontinuous in the sense that every function equivalent to f is everywhere discontinuous. Consider the open unit square intQ. A pole x_0 of a function f is a point such that for each M there is a disc σ with center x_0 such that $|f(x)| > M$ for each $x \in \sigma$. We use the elementary fact that for each $\epsilon > 0$, $k > 0$, and $x \in$ intQ there is a spine function p with center x and height k such that $a(p) < \epsilon$. Now, let $\{x_n: n=1,2,\ldots\}$ be a dense subset of IntQ. For each n let p_n be a sum of spine functions whose centers are $\{x_1,\ldots,x_n\}$, whose heights are all n, and the sum of whose perimeters are less than $1/n2^n$ with $a(p_n) < 1/2^n$. The function $f = \Sigma p_n$ is in W_1^1 and has area less than 2.

e. We turn now to an exposition of some associated results of Federer. There are certain geometric properties of bounded variation functions of one variable which have suitable analogues for functions in BVC. Federer exploits these analogues in his work which is described briefly below. Let $f \in$ BVC. The role of continuity is assumed by approximate continuity and the notion of finite sets (empty set) is replaced by the notion of finite (zero) Hausdorff (n-1) dimensional measure, H_{n-1}. (The zero dimensional measure of a set is equal to its cardinal number). The graph of f is completed by a vertical line segement, ℓ_x, joining the upper and lower approximate limits of f at x for every $x \in Q$. The main results of Federer [2] are listed below as Theorems F_1, F_2, F_3, and F_4.

THEOREM F_1. *If* $f \in$ BVC *then* $E(f)$ *is equal to the Hausdorff* n *dimensional measure of the graph of* f

$$S_f = \cup[\ell_x: x \in Q].$$

THEOREM F_2. *If* $f \in$ BVC, *the variation measure of the vector valued measure determined by grad*(f) *is given by*

$$\int_{-\infty}^{+\infty} H_{n-1}(f^{-1}(y))dy.$$

THEOREM F_3. *If* $f \in$ BVC, *then* f *is approximately continuous except on a set* S *which is the union of countably many sets of finite Hausdorff* (n-1) *dimensional measure*.

THEOREM F_4. *If* $f \in$ BVC, *then at each point, except for a set of Hausdorff* (n-1) *dimensional measure zero, there is a hyperplane though the point on each side of which* f *has a unique approximate limit*.

f. We culminate this section with a statement of Michael's Theorem. We first note that if $f \in$ BVC then f defines an area measure E_f on the Borel subsets of Q.

THEOREM M. *If* $f \in$ BVC *then for each* $\epsilon > 0$ *there is a* $g \in C^1$ *such that* $f(x) = g(x)$ *except on a set* G *with* $m(G) < \epsilon$ *and* $|E_f(Q) - E_g(Q)| < \epsilon$.

In the case that $f \in W_1^1$ it follows immediately that for every $\epsilon > 0$ there is a $g \in C^1$ such that if $G = \{x: f(x) \neq g(x)\}$, then $E_f(G) < \epsilon$ and $E_g(G) < \epsilon$.

§3.

We now discuss the analogue in several variables of the fact that a function of bounded variation is the difference of two nondecreasing functions. Let $f\colon Q \to R$ belong to BVC. We may suppose that f is of bounded variation in x_1 for almost all of the other variables $x_2,\ldots,x_n$. For each such $x_2,\ldots,x_n$ consider the increasing part of f in x_1 as for functions of one variable. It is not difficult to show that the resulting function g of n variables is measurable and summable on Q. Then $f = g - h$, where g and h are increasing in x_1 for almost every $x_2,\ldots,x_n$ and are summable. This is a very useful fact.

It is somewhat surprising that it may be impossible to choose the functions g and h so that they belong to BVC. In the two dimensional case this is a consequence of the following fact which is of interest in its own right.

THEOREM. *If* $f \in$ BVC *is a function of two variables* x *and* y *and* f *is monotonically nondecreasing as a function of* y *for almost all values of* x, *then* f *has an equivalent* g *which is continuous almost everywhere*.

The proof of this theorem rests on two simple lemmas which involve the notion of m-continuity. A function f is *m-continuous* at a point (x,y) if there is a set of measure zero such that f is continuous at (x,y) relative to the complement of that set.

LEMMA 1. _A function f is m-continuous almost everywhere if and only if there is a set S, whose complement has measure zero, such that f is continuous on S relative to S._

Proof. The sufficiency is obvious. For the converse, let T be the set of points of m-continuity and let $\epsilon > 0$. Every point in T is the center of a ball of radius as small as we please in which the oscillation of f, ignoring a set of measure zero, is less than ϵ. By the Vitali covering theorem, there is a set S_ϵ, whose complement is of measure zero, such that the oscillation of f, relative to S_ϵ, is less than ϵ at each point of S_ϵ. Let $S = \bigcap_n S_{1/n}$.

LEMMA 2. _A function f is m-continuous almost everywhere if and only if there is an equivalent function g which is continuous almost everywhere._

Proof. Again, sufficiency is obvious. Suppose f is m-continuous almost everywhere. Let S be a set, whose complement has measure zero, such that f restricted to S is continuous at each point of S. Let

$$g(x) = \limsup_{t \to x, t \in S} f(t).$$

Then $f(x) = g(x)$ on S. Let $x \in S$ and let $\epsilon > 0$. Then there is an $r > 0$ such that if $y \in S$ and $|x-y| < r$, then $|f(x)-f(y)| < \epsilon$. Hence, for any y such that $|x-y| < r$, we have $|f(x)-f(y)| \leq \epsilon$ and, consequently, g is continuous at each $x \in S$.

Proof of the **theorem.** We suppose the support of f is in the unit square. We shall show that f is m-continuous almost everywhere. Suppose, on the contrary, that the set of points where f is not m-continuous has positive measure. We may then assume that there is a $k > 0$ and a measurable set E such that $m(E) > k$ and the upper m-limit of f at (x,y) exceeds $f(x,y)$ by more than k. Since f is continuous in y almost everywhere, we may assume that this holds at every point in E. For each x_0, let $E_{x_0} = \{(x,y) \in E: x = x_0\}$. Then $m(E_x) > k$ for an infinite set of values of x. Let r be a positive integer and let $x_1 < \ldots < x_r$ be such that $m(E_{x_i}) > k$, $i = 1,\ldots, r$. Let $a = \min\{(x_i-x_{i-1}): i = 2,\ldots,r\}$. Then a is positive. Let i be an integer between 1 and r, let $(x_i,y_0) \in E_{x_i}$, and let $\epsilon > 0$. There is an h such that $0 < h < \epsilon^i$ and $f(x_i,y_0+h) - f(x_i,y_0) < k/2$. There is a (u,v) with $|x_i-u| < a/2$, $v < y_0 + h/2$, $f(u,v) > f(x_i,y_0) + k$, and with f monotonically decreasing as a function of y for $x = u$. Now, $f(u,y) > f(x_i,y) + k/2$ for a set of values of y whose relative measure in $[y_0,y_0+h]$ exceeds 1/2. By the Vitali covering theorem there are pairwise disjoint intervals $I_{i1},\ldots, I_{in_i}$ on the line $x = x_i$, the sum of whose lengths exceeds k/2 and such that for each $j = 1, \ldots n_i$, there is an a_j with $|a_j| < a/2$ such that for every $y \in I_{ij}$

$$f(x_i+a_j,y) - f(x_i,y) > k/2.$$

If $V_f(y)$ is the variation of f as function of x, for y fixed, then the above inequality implies that $\int_0^1 V_f(y)dy > k^2r/2$ and since this holds for every r, $V_f(y)$ is not summable. This contradiction completes the proof.

Now, suppose f is an everywhere discontinuous BVC function. Let f = g-h be a decomposition of the type discussed in the opening paragraph of this section. The functions f and g cannot both be in BVC, for if they were, the above theorem would dictate that f be continuous almost everywhere.

By a rather complex discussion one may use this two variable result to further show that there is a BVC function f of n variables such that for any direction d, f fails to have any decomposition f = g-h for which both g and h are in BVC and are nondecreasing in the direction d.

The theorem of this section does not hold for $n > 2$. This depends on an example, a refinement of one given in the previous section, of a function g of n-1 variables which is in the Sobolev space W^1_{n-1} and is everywhere discontinuous. Letting $f(x_1,\ldots,x_n) = g(x_2,\ldots,x_n)$, we have an example of an everywhere discontinuous function of n variables which belongs to W^1_{n-1} and which is monotonically nondecreasing in x_1 for almost all values of the other variables. We point out, however, that if we have a function in W_p^1, $p > n-1$, then the analogue of our theorem does hold.

§ 4.

We now define a special class which lies between W_1^1 and BVC and which plays the role in n variables that the continuous bounded variation functions play in one variable. This class is denoted by L and as the n-dimensional case is wholly analogous to the two variable case, we will restrict our attention to two dimensions. The material of this section and the next can be found in [5] and [6].

Let Q be the closed unit square. A function $f: Q \to R$ is called <u>essentially linearly continuous</u> if f has equivalent functions g and h such that g is continuous in y for almost every x, and h is continuous in x for almost every y. The function f is <u>linearly continuous</u> if it is continuous in each variable for almost all values of the other variable. Finally, the function f is <u>strongly linearly continuous</u> if for each direction it is continuous as a function of one variable along almost every line in that direction.

THEOREM. *If f is essentially linearly continuous and BVC then there is an equivalent function g which is linearly continuous.*

We will give some indication of the proof. Let $\{\varphi^h : h > 0\}$ be a family of regularizers and let $f^h = \varphi^h * f$. As f is essentially linearly continuous, there is an equivalent function g which is continuous in y for almost all values of x. As f is BVC, $g = g^+ - g^-$ where g^+ and g^- are continuous and monotonically nondecreasing in y for almost every x and such that each of g^+ and g^- is summable. Now, $(g^+)^h$ converges to g^+ almost everywhere, and $(g^+)^h$ is monotonically nondecreasing in y for every x. So, for almost every x, the monotonically nondecreasing $(g^+)^h$ converges almost everywhere to the montonically nondecreasing g^+. It is easy to see that for such x the convergence is everywhere and uniform. Analogous results hold for $(g^-)^h$ and g^-.

It follows that g^h converges uniformly on almost all lines parallel to one of the axes. If the φ^h are symmetric (i.e. are square integral means) then g^h also converges uniformly on almost all lines parallel to the other axis. The function lim sup g^h is equivalent to f and is linearly continuous.

Now, it is also true that f is equivalent to a strongly linearly continuous function, but this result is somewhat deeper and is the topic of the next section.

Let $Q = I\text{x}J$. If f is linearly continuous, then for each $\varepsilon > 0$ there are closed sets $E \subseteq I$ and $F \subseteq J$ such that the measure of each set is at least $1 - \varepsilon$ and such that f is continuous on $(E\text{x}J) \cup (I\text{x}F)$. It follows that the space of linearly continuous functions is complete with respect to the metric $d(f,g)$ where

$$d(f,g) = \inf\{k\colon \text{there are } E \subseteq I,\ F \subseteq J \text{ with } |E| > 1-k \text{ and } |F| > 1-k \text{ such that } |f(x,y) - g(x,y)| < k \text{ for each } (x,y) \in (E\times J) \cup (I\times F)\}.$$

The piecewise linear functions are dense in this space. The area functional, $a(p)$, is lower semicontinuous with respect to this metric and extends to $E(f)$ on the entire space. The intersection of BVC with this space is the space, L, of linearly continuous functions of finite area.

Federer [3] (also Mickle and Rado [13]) has shown that if f is continuous, then $A(f) = H^2(gr(f))$, [$H^2 \equiv$ Hausdorff two dimensional measure; $gr(f) \equiv$ the graph of f]. It follows easily that the area measure, $A_f(S)$, is equal to the Hausdorff measure of the graph of f restricted to S. This result extends to the functions in L; sets whose projections on both axes have measure zero are negligible and have E_f measure zero. In computing Hausdorff measure, such a set Z is ignored.

THEOREM. <u>If</u> $f \in L$, <u>then its surface area is equal to the Hausdorff two dimensional measure of the graph of</u> f <u>on</u> $Q\setminus Z$.

We also have a Verchenko type theorem, i.e.

THEOREM. *If* $f, g \in L$ *and* $f \equiv g$ *on the Borel set* S, *then* $E_f(S) = E_g(S)$.

We conjecture that the functions in L are those whose gradient measures are zero on sets whose (n-1) Hausdorff measure is finite.

§ 5.

In this section we show that functions in L are strongly linearly continuous for n=2. The cases where $n > 2$ are much more difficult and will be considered later. First, however, we give the following example which shows that BVC is a necessary hypothesis in the result mentioned above.

EXAMPLE. *There is a linearly continuous function* f *which is not strongly linearly continuous.*

Proof. Let E denote the Cantor ternary set. Then the projection of ExE onto the line y=x is a line segment of length $\sqrt{2}$. Let J_n be the union of the 2^{n-1} open intervals of length 3^{-n} which are contiguous to E. Let $I_n = [0,1] \bigcup_{n=1}^{n} I_n$ and let $I_0 = [0,1]$. Now,

$$Q \setminus E \times E = \bigcup_{n=1}^{\infty} C_n \text{ where}$$

$$C_n = (J_n \times I_{n-1}) \cup (I_{n-1} \times J_n)$$

the sets C_n are the n^{th} stage "crosses" contiguous to ExE. Shrink each interval in J_n to a concentric closed interval and let J_n' denote the union of these closed but slightly smaller intervals. Finally, let

$$K_n = (J_n' x I_{n-1}) \cup (I_{n-1} x J_n')$$

and define the function $f: (\bigcup_{n=1}^{\infty} K_n) \cup (ExE) \to [0,+\infty]$ by

$$f(x) = \begin{cases} n \text{ if } x \in K_n \\ +\infty \text{ if } x \in ExE \end{cases}$$

It is easy to see that f is continuous on $(\bigcup_{n=1}^{\infty} K_n) \cup (ExE)$ and so has a continuous extension $f_*: Q \to [0,+\infty]$ such that $f_*^{-1}(+\infty) = ExE$. If we consider f_* as a map from Q into R (redefining $f_*(ExE) = -1$) then f_* is continuous in y for each $x \notin E$. However, if ℓ is any line meeting intQ then ℓ contains a point of ExE and consequently f_* is not continuous on ℓ. Further, if g is any function equivalent to f, then g=f on almost all of each cross K_n. It follows that g is also discontinuous on almost every diagonal line which intersects intQ. This completes the presentation of the example.

We now introduce some preliminary work which will be used in the proof of the main result of this section. We will consider specific integral means f^h where

$$f^h(x,y) = \frac{1}{\pi h^2} \int_{S((x,y),h)} f(u,v)\,dudv.$$

Let $f \in BVC$, then f^h is continuous and $E(f|I) = \underset{h \to 0}{\text{limit}}(f^h | I_{-h})$.

Here, I_{-h} denotes the interval which is concentric with I, but whose side lengths are each 2h shorter than the respective side lengths of I. In addition we have the following lemma which we state without proof.

LEMMA. <u>If</u> $f \in L$ <u>and</u> I <u>is any open interval, then</u> $E(f|I) = \lim_{h\to 0} E(f^h|I)$.

Now, suppose $f \in L$ and let $G = (a,b)x(c,d)$ contain the support of f. Let $\epsilon > 0$ and let $S\subset(a,b)$ and $T\subset(c,d)$ be compact sets such that:

i. For each $x_0 \in S$, $f^h(x_0,y)$ converges uniformly on (c,d) to $f(x_0,y)$, and, for each $y_0 \in T$, $f^h(x,y_0)$ converges uniformly on (a,b) to $f(x,y_0)$.

ii. The function f is uniformly continuous on $M = Sx(c,d) \cup (a,b)xT$.

iii. $E_f(M) > E_f(G)-\epsilon$

We can define a function in each of the open intervals contiguous to M in such a way that this function is continuous, agrees with f on the boundary of the domain interval and has small area. The actual argument is rather delicate, but it does yield the following result.

THEOREM. <u>If</u> $f \in L$ <u>and</u> $\epsilon > 0$, <u>there is a continuous function</u> g <u>such that if</u> $S = \{(x,y): f(x,y) \neq g(x,y)\}$ <u>then</u> $E_f(S) < \epsilon$ <u>and</u> $E_g(S) < \epsilon$.

The converse of this theorem is also true. Let $f \in BVC\setminus L$. There is a set $K\subset(a,b)$ of positive measure such that for every $x_o \in K$ there is a y_o and a $k>0$ with the properties that if $y \in (y_o-k, y_o)$ and $y' \in (y_o, y_o+k)$ then $f(x_o,y') > f(x_o,y) + k$ (this is but one of four possibilities). It follows that there is a $u>0$ and an $S \subset (a,b)$ with $m(S)>u$ and such that for every $x_o \in S$ there is a $y_o \in (c,d)$ for which $y \in (y_o-u,\ y_o) \equiv I_{x_o}$. Also, $y' \in (y_o, y_o+\ ) = J_{x_o}$ implies that $f(x_o,y') > f(x_o,y) + u$. Let $\varepsilon = u^2/4$, and suppose g is continuous with $E_f(I) < \varepsilon$ where $T = \{(x,y)\colon f(x,y) \neq g(x,y)\}$. Then, $m(T) < E_f(T) + \varepsilon$. There is then a $y \in I_{x_o}$ such that $f(x_o,y) = g(x_o,y)$ for every $x_o \in S$ except for a subset whose measure is less than $u/4$. Similarly, there is a $y' \in J_{x_o}$ such that $f(x_o,y') = g(x_o,y')$ for every $x_o \in S$ except for a subset of measure less than $u/4$. It follows that there is a $U \subset S$ with $m(U)>u/2$ such that for every $x_o \in U$ there is a $y \in I_{x_o}$ and a $y' \in J_{x_o}$ for which $f(x_o,y) = g(x_o,y)$ and $f(x_o,y') = g(x_o,y')$. For each $x_o \in U$, let H_{x_o} be the maximal closed interval containing (x_o,y_o) in its interior on which $f(x_o,y) \neq g(x_o,y)$ except possibly at (x_o,y_o) and at the endpoints. The variation of g on this interval exceeds u and as such, $m_1(T) > u^2/2 > \varepsilon$ where m_1 is the partial derivative of g with respect to x. But, $E_g(T) \geq m_1(T)$ and the theorem is proved.

As a corollary we now obtain the desired result.

COROLLARY. *If* f *is essentially linearly continuous and* BVC, *then* f *is strongly linearly continuous*.

§ 6.

The goal of the final three sections of this survey is to show that, in any dimension n, a BVC function $f: Q \to R$ belongs to L if and only if, for each $\epsilon>0$, there is an approximately continuous function g such that if $E = \{x: f(x) \neq g(x)\}$, then $\alpha_f(E) < \epsilon$ and $\alpha_g(E) < \epsilon$, where $\alpha_f(E)$ denotes the area measure determined by f, [7] and [8]. This area measure α_f may be obtained in two ways: first, we may extend the area from an interval function to one defined on the Borel sets; or if λ is Lebesgue measure and $(\mu_1, \ldots, \mu_n)$ is the gradient measure corresponding to the derivative of f, then α_f is the variation measure of the vector measure $(\lambda, \mu_1, \ldots, \mu_n)$. The so-called co-area or partial area is the variation measure β_f of $(\mu_1, \ldots, \mu_n)$. As an outer measure we have that α_f satisfies the following three conditions:

(i) if the distance between E and F is positive, then $\alpha_f(E \cup F) = \alpha_f(E) + \alpha_f(F)$;

(ii) if E is measurable and $\epsilon>0$, there is a compact F and an open G with $F \subset E \subset G$ and $\alpha_f(G) < \alpha_f(F) + \epsilon$;

(iii) compact sets have finite measure.

The following Vitali-Besicovitch covering theorem will be useful to us.

THEOREM. _If an outer measure_ m _on n-space satisfies the above conditions_ (i), (ii), (iii) _and if_ S _is a set covered by a family_ R _of closed oriented cubes such that_ *for each $\epsilon>0$ and $x \in S$ there is a cube in R with center x and diagonal less than ϵ, then a countable set of pairwise disjoint cubes in R covers almost all of S.*

We shall let f_m denote the spherical integral means of f over balls of radius $1/m$. The sequence $\{f_m\}$ converges uniformly along almost every line parallel to the coordinate axes to a function g which is equivalent to f. Also, for each $\delta>0$ there is a set $E_\delta = \bigcup_{i=1}^{n}(E_i \times (a_i,b_i))$ on which g is uniformly continuous and on which $f_m = g_m$, $m = 1,2,\ldots$ converges uniformly to g, and $\lambda(Q_i\ E_i) < \delta$, $i = 1,\ldots,m$. We can say even more:

LEMMA. *Using the notation of the previous paragraph, given an $\epsilon>0$ we can choose E_δ for which $\alpha_f(Q\setminus E_\delta) < \epsilon$.*

Proof. For each $i=1,\ldots,n$,

$$|\mu_i|(Q) = \int_{Q_i} V(f,\overline{x}_i,(a_i,b_i))d\overline{x}_i,$$

where $\overline{x}_i$ denotes the vector in n-1 space obtained by deleting x_i from $(x_1,\ldots,x_n)$. Now, there is a $\delta>0$, $\delta < \epsilon/(n+1)$, such that $\lambda(Q_i\setminus E_i) < \delta$ implies

$$|\mu_i|(E_i x(a_i,b_i)) = \int_{E_i} V(f,\overline{x}_i,(a_i,b_i))d\overline{x}_i > |\mu_i|(Q) - \epsilon/(n+1).$$

For $E_\delta = \bigcup_{n=1}^{n}(E_i x(a_i,b_i))$ we have

$$\alpha_f(Q\setminus E_\delta) \le \sum_{i=1}^{n} |\mu_i|(Q\setminus E_\delta) + \lambda(Q\setminus E_\delta)$$

$$\le \sum_{i=1}^{n} |\mu_i|\{(Q_i\setminus E_i) \times (a_i,b_i)\} + \lambda(Q\setminus E_\delta)$$

$$= \sum_{i=1}^{n} |\mu_i|(Q) - \sum_{i=1}^{n} |\mu_i|(E_i x(a_i,b_i)) + 2(Q\setminus E_\delta)$$

$$< \epsilon.$$

In §7 we shall verify the necessity of the condition mentioned in the beginning of the present section and in §8 we shall prove its sufficiency. Before giving the necessity proof we consider the following construction which will be used in that argument. Let $f \in L$ and let $\epsilon>0$. We obtain a special zero dimensional closed set V for which $\alpha_f(V) > \alpha_f(Q) - \epsilon$.

To this end let E_δ be as in the previous lemma. For each $i=1,\dots,n$, let $S_i \subset E_i$ be the points at which the $(n-1)$ density of E_i using cubes in Q_i is 1. Let $S = \bigcup_{i=1}^{n} (S_i x(a_i,b_i))$. Then $\alpha_f(Q\setminus S) < \epsilon$. Let $T_i \subset S_i$ be closed and such that $\lambda(Q_i\setminus T_i) < \delta$. If $T = \bigcup_{i=1}^{n} (T_i x(a_i,b_i))$ then $\alpha_f(Q\setminus T) < \epsilon$ and T is closed in Q.

We apply the above covering theorem to T to obtain a system of oriented closed cubes. Let R_x be a collection of oriented closed cubes centered at x, all of diagonal less than 1/2, and some arbitrarily small. Let

$$R = \cup\{R_x : x \in \operatorname{int} Q \cap T\}.$$

There is a finite set $R_1,\ldots,R_{n_1}$ of pairwise disjoint cubes in R such that

$$\alpha_f\{Q\setminus \bigcup_{j_1=1}^{n_1}(R_{j_1}\cap T)\} < \varepsilon.$$

In the same way, for each $j_1=1,\ldots,n_1$, there is a finite set $R_{j_1 1},\ldots,R_{j_1 n_{j_1}}$, pairwise disjoint, interior to R_{j_1}, such that

$$\alpha_f\{Q\setminus \bigcup_{j_1=1}^{n_1}\ \bigcup_{j_2=1}^{n_{j_1}} (R_{j_1 j_2}\cap T)\} < \varepsilon .$$

Continuing, we obtain a system

$R_{j_1\cdots j_k}$, $k = 1,2,\ldots$

$j_1 = 1,\ldots,n_1$

for each j_1, $j_2 = 1,\ldots,n_{j_1}$

for each $j_1 = 1,\ldots,n_1$ and $j_2 = 1,\ldots,n_{j_1}$;

$j_3 = 1,\ldots,n_{j_1 j_2}$;

⋮

for each $j_1,\ldots,j_{k-1}$; $j_k = 1,\ldots,n_{j_1\cdots j_{k-1}}$,

of pairwise disjoint closed cubes. Each $R_{j_1\cdots j_{k-1}j_k}$ is in $\operatorname{int}R_{j_1\cdots j_{k-1}}$ and is of diagonal less than $1/2^k$. For each k the cubes $R_{j_1\cdots j_k}$ are pairwise disjoint and

$$\alpha_f(Q \cup(R_{j_1\ldots j_k}\cap T)) < \varepsilon.$$

The $R_{j_1 \cdots j_k}$ are said to be of rank k.

For each k=1,2,..., let $V_k = \cup(R_{j_1 \cdots j_k} \cap T)$ and let $V = \bigcap_k V_k$. Since T is closed it follows that

$$V = \bigcap_{k=1}^{\infty} (\cup R_{j_1 \cdots j_k}).$$

In this construction all cubes chosen are taken so that $\{f_m\}$ converges uniformly on almost all lines on their faces which are parallel to the coordinate axes.

The density of S is 1 at each point of T and hence of V. We consider a closed frame about the boundary of each $R_{j_1 \cdots j_k}$ so that the frames are pairwise disjoint and their union has density zero at each point of V. This is done as follows: For any cube R we consider frames of width h about ∂R. Each $R_{j_1 \cdots j_k}$ has a frame $G_{j_1, \ldots, j_k}$ and they may be taken so that they are all pairwise disjoint, even of different ranks. Each $G_{j_1 \cdots j_k}$ may then be shrunk to a frame $F_{j_1 \cdots j_k}$, still about $\partial R_{j_1 \cdots j_k}$, in such a way that $\cup\cup F_{j_1 \cdots j_k}$ has density zero at each $x \in V$. Then $S \setminus \cup\cup F_{j_1 \cdots j_k}$ has density one at each $x \in V$.

We make one further remark at this point. If $f \in L$ and H is an oriented hyperplane, then $\alpha_f(H)=0$. So, for any open interval I it readily follows that $\lim_m \alpha_{f_m}(I) = \alpha_f(I)$. This may fail for $f \in BVC$.

§7.

In this section we complete the proof of the following theorem.

THEOREM. <u>For each</u> $f \in \mathcal{L}$ <u>and</u> $\varepsilon > 0$ <u>there is an approximately continuous</u> g <u>such that if</u> $E = \{x\colon f(x) \neq g(x)\}$, <u>then</u> $\alpha_f(E) < \varepsilon$ <u>and</u> $\alpha_g(E) < \varepsilon$.

Proof. Let V be as in the previous section. We shall extend f, restricted to V, to a function g on Q which is approximately continuous at every $x \in Q \setminus V$, except possibly at points on the boundaries of the cubes $R_{j_1 \ldots j_k}$.

Via a truncation of f, which reduces the area slightly, we may suppose $|f(x)| < M$ for each $x \in Q$.

For each k, let $Q_k = \cup R_{j_1 \ldots j_k}$ and let A_k be the $(n-1)$ measure of ∂Q_k. We choose sequences $\{\eta_k\}$, $\{\zeta_k\}$, $\{\xi_k\}$, where $\Sigma \xi_k < \varepsilon$, $\Sigma \eta_k A_k < \varepsilon$ and $\Sigma \zeta_k < \varepsilon / 2M$.

We first define g on $Q \setminus Q_1$ as an integral mean f_{m_1} so that

(a_1) $|f_{m_1}(x) - f(x)| < 1$ for each $x \in S$ (see §6),

(b_1) $|f_{m_1}(x) - f_m(x)| < \eta_1$ on all of ∂Q_1 except for a set of $(n-1)$ measure less than ζ_1 for each $m > m_1$,

(c_1) $\alpha_{f_{m_1}}(Q \setminus \operatorname{int} Q_1) < \alpha_f(Q \setminus \operatorname{int} Q_1) + \xi_1$.

Now, (c_1) follows from the last remark in the previous section and (a_1) and (b_1) follow from the choice of S and the properties of the R_{j_1}.

We next define g on $Q_1 \setminus \mathrm{int} Q_2$ as f_{m_2}, where $m_2 > m_1$ is such that

(a_2) $|f_{m_2}(x) - f(x)| < 1/2$ on S,

(b_2) $|f_{m_2}(x) - f_m(x)| < \eta_2$ on all of ∂Q_2 except for a set of $(n-1)$ measure less than ζ_2 for each $m > m_2$,

(c_2) $\alpha_{f_{m_2}}(Q_1 \setminus \mathrm{int} Q_2) < \alpha_f(Q_1 \setminus \mathrm{int} Q_2) + \xi_2$.

Continuing in this manner we obtain $m_1 < m_2 < \ldots < m_k < \ldots$ and for each k we define g on $Q_{k-1} \setminus \mathrm{int} Q_k$ as f_{m_k} where

(a_k) $|f_{m_k}(x) - f(x)| < 1/k$ on S,

(b_k) $|f_{m_k}(x) - f_m(x)| < \eta_k$ on ∂Q_k for each $m > m_k$ except on a subset of measure less than ζ_k,

(c_k) $\alpha_{f_{m_k}}(Q_{k-1} \setminus \mathrm{int} Q_k) < \alpha_f(Q_{k-1} \setminus \mathrm{int} Q_k) + \xi_k$.

Since $Q\ V = \bigcup_{k=1}^{\infty} Q_{k-1} \setminus \mathrm{int} Q_k$, where $Q_0 = Q$, we have defined g everywhere on Q. It is two branched on ∂Q_k. Pick one of the branches.

We now remove the discontinuities on the ∂Q_k sets. Note that any modification of g within the frames $F_{j_1 \ldots j_k}$ will not change the approximate continuity of g at any point of V. Let $R = R_{j_1 \ldots j_k}$ be any cube in our system and F the frame about ∂R. We shall transfer the discontinuities of g from ∂R to the surfaces of two other cubes with boundaries in F, but with greatly reduced oscillations. By repeating the process, we obtain a sequence of functions which converge to a continuous h in F with $\alpha_h(F)$ not much greater than $\alpha_g(F)$.

To do this, let $\eta > 0$, $\Sigma \xi_k < \eta$, $\Sigma \zeta_k < \infty$. Let A and B be closed cubes concentric with R with ∂A and ∂B in intF, $A \subset \text{int}R$, $R \subset \text{int}B$. Then let $\{A_m\}$ be a decreasing sequence and $\{B_m\}$ an increasing sequence of closed cubes concentric with R, converging to A and B, respectively, with $A_1 \subset \text{int}R$, $R \subset \text{int}B_1$, $\text{int}A_m \supset A$, $\text{int}B \subset B_m$, m=1,2,... and $A_{m+1} \subset \text{int}A_m$, $B_m \subset \text{int}B_{m+1}$, m=1,2,... . Let H_m be the frame bounded by ∂A_m and ∂B_m and H the one bounded by ∂A and ∂B. Then $\{H_m\}$ is an increasing sequence of frames about ∂R all contained in H, which is in turn in intF.

We define a sequence $\{g^m\}$ of functions on F.
Let $g^1 = g$ on $F \setminus H_1$. On H_1 let $g^1 = g_{k_1}$ where k_1 is so large that

(α_1) $|g(x) - g_{k_1}(x)| < \zeta_1$ on ∂H_1,

(β_1) $\alpha_{g^1}(\text{int}H_1) < \alpha_g(\text{int}H_1) + \xi_1$,

(γ_1) $\alpha_{g^1}(\partial H_1) < \alpha_g(\partial H_1) + \xi_1$, and

(δ_1) $|g^1(x) - g(x)| < 2K$ for every $x \in F$ where K is the maximum of the oscillation of g on ∂R.

Then $\alpha_{g^1}(F) < \alpha_g(F) + 2\xi$, and the oscillation of g^1 does not exceed ζ_1 at any point of F.

Next, let $g^2 = g^1$ on $F \setminus H_2$. On H_2 let $g^2 = g^1_{k_2}$ where k_2 is such that

(α_2) $|g^1(x) - g^1_{k_2}(x)| < \zeta_2$ on ∂H_2,

(β_2) $\alpha_{g^2}(\text{int}H_2) < \alpha_{g^1}(\text{int}H_2) + \xi_2$,

(γ_2) $\alpha_{g^2}(\partial H_2) < \alpha_{g^1}(\partial H_2) + \xi_2$, and

(δ_2) $|g^2(x) - g^1(x)| < 2\zeta_1$ for each $x \in F$.

Then $\alpha_{g^2}(F) < \alpha_{g^1}(F) + 2\xi_2$ and the oscillation of g^2 does not exceed ζ_2 at any point of F.

Continuing we obtain a sequence $\{g^m\}$ of functions each continuous and agreeing with each other on $F \setminus H$ and such that

(i) g^m is discontinuous only on ∂H_m,

(ii) $|g^{m+1}(x) - g^m(x)| < 2\zeta_m$ for each $x \in F$

(iii) $\alpha_{g^m}(F) < \alpha_{g^{m-1}}(F) + 2\xi_m$,

(iv) the oscillation of g^m does not exceed ζ_m at any point of F.

Now, $\{g^m\}$ converges uniformly to a function h. For each m and each $x \in F$ the oscillation of h at x is less than

$$t_m = \zeta_m + \sum_{r=m}^{\infty} 2\zeta_r.$$

Since $\{t_m\}$ is a null sequence, h is continuous on F.

By lower semicontinuity

$$\alpha_h(F) \leq \liminf \alpha_{g^m}(F) < \alpha_g(F) + 2\sum_{m=1}^{\infty} \xi_m < \alpha_g(F) + 2\eta \, .$$

Having established this, we order and relable the countable set of frames $F_{j_1 \ldots j_k}$ as $F_1, F_2, \ldots, F_m, \ldots$. Let $\eta_m > 0$ be such that $\sum_{m=1}^{\infty} \eta_m < \epsilon$. For each m modify g as above in F_m so that

$$\alpha_h(F_m) < \alpha_g(F_m) + \eta_m.$$

Then α_h has the desired properties and the theorem is proved.

§ 8.

We now treat the converse. Let $f \in BVC$ and for a direction $i = 1,\ldots,n$, let Q_i denote the n-1 cube obtained by deleting the i^{th} coordinate from Q. So, $Q = Q_i x(a,b)$. For each $x_i \in (a,b)$ and $\overline{x}_i \in Q_i$, let $x = (x_i,\overline{x}_i)$ and let $V(x) = V(x_i,\overline{x}_i)$ be the variation of $u(x_i,\overline{x}_i)$ on the interval (a,x_i), where u is the upper linear measurable boundary of f. Then V(x) is measurable.

Suppose $f \notin L$. For reals $0 < s < t$ let

$$T = \{x\colon V(x)<s\} \text{ and } U = \{x\colon V(x)>t\}.$$

These sets are measurable. Define functions φ and ψ on Q_i as follows:

$$\varphi(\overline{x}_i) = \sup\{x_i\colon (x_i,\overline{x}_i)\in T\}$$

$$\psi(\overline{x}_i) = \inf\{x_i\colon (x_i,\overline{x}_i)\in U\}.$$

These functions are measurable. Since $V(x) = V(x_i,\overline{x}_i)$ is monotonically non-decreasing in x_i for each $\overline{x}_i$ and for some i the set $\{\overline{x}_i | V \text{ is discontinuous at } (x_i,\overline{x}_i) \text{ for some } x_i\}$ is not of measure zero, there is an (s,t) for which the corresponding φ and ψ are such that the set

$$A = \{\overline{x}_i\colon \varphi(\overline{x}_i)=\psi(\overline{x}_i)\}$$

has positive measure.

We need the following

LEMMA. *If* f *is a measurable function of* n *variables on a measurable set* A, *then for every* $k > 0$, *for almost all* $x \in A$, *the upper densities at* x *of the sets for which*

$$\frac{f(y)-f(x)}{|y - x|} > -k \text{ and } \frac{f(y)-f(x)}{|y - x|} < k$$

are positive.

From these facts we obtain by a delicate computation the following

THEOREM. *If* $f \in BVC$ *is approximately continuous everywhere, then* $f \in \mathcal{L}$.

Another computation yields

LEMMA. *If* $f \in BVC \setminus \mathcal{L}$, *there is a* $k > 0$ *such that for every* $g \in \mathcal{L}, \alpha_g(E) > k$, *where* $E = \{x\colon f(x) \neq g(x)\}$.

These two facts complete the proof of

THEOREM. *A function* $f \in \mathcal{L}$ *if and only if for each* $\epsilon > 0$ *there is an approximately continuous* g *such that* $\alpha_f(E) < \epsilon$ *and* $\alpha_g(E) < \epsilon$ *where* $E = \{x\colon f(x) \neq g(x)\}$.

This theorem implies that if $f \in \mathcal{L}$, then f is strongly linearly continuous. This follows from the fact that if f is approximately continuous and g is bilipschitzian, then fog is linearly continuous. In particular bilipschitzian maps take d open sets (open sets in the density topology) into d open sets.

We now put a metric on L and observe that both L and its subspace W_1^1 are complete. Let $\delta(f,g)$ be the convergence in measure metric and $\Delta(f,g)$ the supremum of $|\alpha_f(E)-\alpha_g(E)|$ where $E = \{x: f(x)\neq g(x)\}$. Then let d be the metric on L given by

$$d(f,g) = \delta(f,g) + \Delta(f,g).$$

THEOREM. L <u>is complete</u>.

Proof. Suppose $f \in BVC\setminus L$. Then for some $i=1,\ldots,n$, for a set E of positive outer measure there is a $k > 0$ and $\delta > 0$ such that for each $\overline{x}_i \in E$ there is an $x_i(\overline{x}_i)$ such that

$$f(x_i(\overline{x}_i)+, \overline{x}_i) > f(x_i(\overline{x}_i)-, \overline{x}_i) + k$$

and the variation of $f(x_i,\overline{x}_i)$ in x_i is less than $k/8$ in each of the intervals $(x_i(\overline{x}_i)-\delta, x_i(\overline{x}_i))$ and $(x_i(\overline{x}_i), x_i(\overline{x}_i) + \delta)$. There is an $\eta > 0$ such that $\delta(f,g) < \eta$ implies that for each $\overline{x}_i$, not in a set of measure less than $m_e(E)/2$, for every interval $(x_i,x_i+\delta)$, there is a $\xi_i \in (x_i,x_i+\delta)$ such that $|f(\xi_i,\overline{x}_i)-g(\xi_i,\overline{x}_i)| < k/8$.

So, let g be continuous and such that $\delta(f,g) < \eta$. There is $F \subset E$, with $m_e(F) > m_e(E)/2$, such that for every $\overline{x}_i \in F$ there are intervals $I = (x_i(\overline{x}_i)-\delta,x_i(\overline{x}_i))$ and $J = (x_i(\overline{x}_i(\overline{x}_i)), x_i(\overline{x}_i)+\delta)$ on which the sum of the variations of g exceeds $k/2$. Then $\Delta(\alpha_f,\alpha_g) > k\, m_e(E)/4$. Since BVC is complete, so is L.

THEOREM. W_1^1 <u>is complete</u>.

Proof. We shall indicate the proof for the one variable case. Let $f \in L \setminus W_1^1$ with support in (a,b). There is a $k > 0$ and compact E of measure zero such that for each $\delta > 0$ there is a disjoint set of intervals $[a_1,b_1],\ldots,[a_m,b_m]$, the sum of whose lengths is less than δ, with $\sum_{i=1}^{m} |f(b_i)-f(a_i)| > k$. Let $g \in W_1^1$. There is a $\delta > 0$ such that for each set S of measure less than δ, $\alpha_g(S) < k/2$. For $S = \cup[a_i,b_i]$, $\alpha_f(S) > k$, $\alpha_g(S) < k/2$. Now $\alpha_f(A) = \alpha_g(A)$ for any A on which $f(x) = g(x)$. So there is a $B \subset S$ on which $f(x) \neq g(x)$ and $\alpha_f(B) > \alpha_g(B) + k/2$. Hence W_1^1 is complete.

References

[1] L. Cesari, Sulle funzioni a variation limitata, Ann. Sculola Norm. Pisa (2), 5(1936), 219-313.

[2] H. Federer, Geometric Measure Theory, New York, Springer, 1969.

[3] H. Federer, The (φ,k) rectifiable subsets of n space, Trans. Amer. Math. Soc., 62(1947), 114-192.

[4] C. Goffman, Lower semi continuity and area functionals, I. The non-parametric case, Rend. Circ. Mat. Palermo (2), 2(1954), 203-235.

[5] C. Goffman, Decomposition of functions whose partial derivatives are measures, Mathematika 15(1968), 149-152.

[6] C. Goffman, Coordinate invariance of linear continuity, Arch. Rational Mech. Anal., 20(1965), 153-162.

[7] C. Goffman, Non parametric surfaces given by linearly continuous functions, Acta Math., 103(1960), 269-291.

[8] C. Goffman, A characterization of linearly continuous functions whose partial derivatives are measures, Acta Math., 117(1967), 165-190.

[9] C. Goffman and J. B. Serrin, Sublinear functions of measures and variational integrals, Duke Math. J., 31(1964), 159-178.

[10] K. Krickeberg, Distributioners, Funktionen beschränkter Variation, und Lebesquescher Inhalt nichtparametrische Flächen, Ann. Mat. Pura Appl. IV, 44(1957), 105-133.

[11] F. C. Lui, Approximation-extension type properly of continuous functions of bounded variation, J. Math. Mech. 19(1969/70), 207-218.

[12] J. H. Michael, The equivalence of two areas for non-parametric discontinuous surfaces, Illinois J. Math., 7(1963), 59-78.

[13] E. J. Mickle and T. Rado, On a theorem of Besicovitch in surface area theory, Rev. Mat. Univ. Parma 2(1951), 19-45.

[14] L. Tonelli, Sulla quadratura delle superficie, Athi Accad. Naz. Lincei, 6(1926), 357-63.

[15] I. Verchenko, Über das Flachenmas von Mengen, Mat. Sbornik, 10, 52(1942), 11-32.

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