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R. Fleissner, Department of Mathematics, Western Illinois University, Macomb, Illinois 61455.

A Note on Baire 1 Darboux Functions

Let DB_1 denote the class of Baire 1 Darboux functions. For a function f to have the property that f+g beongs to DB_1 for each g in DB_1, it is necessary and sufficient that f be continuous. A proof of this is given in [1, p. 12] and the analogous question for multiplication is raised. Let $M(DB_1)$ denote the class of functions f such that $g\in DB_1$ implies $fg\in DB_1$. It is shown in [1] that continuity is a sufficient, but not a necessary condition for membership in $M(DB_1)$. Sufficiency follows easily from Young's criterion for a Baire 1 function to have the Darboux property [2]:

<u>If</u> f <u>belongs to</u> B_1, <u>then</u> f <u>belongs to</u> DB_1 <u>if and only if at each point</u> x <u>there exist sequences</u> $x_n \uparrow x$ <u>and</u> $y_n \downarrow x$ <u>such that</u> $f(x)=\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} f(y_n)$.

From this criterion it also follows that the function $f(x)=\sin(1/x)$, $f(0)=0$ belongs to $M(DB_1)$ since $f(0)g(0)=0$ and $f(x_n)g(x_n)=0$ where $f(x_n)=0$. The following characterization of $M(DB_1)$ shows that this type of discontinuity is the only kind a member of $M(DB_1)$ can have.

Theorem. <u>A function</u> f <u>belongs to</u> $M(DB_1)$ <u>if and only if</u> (i) f <u>belongs to</u> DB_1 <u>and</u> (ii) <u>if</u> f <u>is discontinuous from the right</u> (<u>resp. left</u>) <u>at</u> x=a, <u>then</u> f(a)=0 <u>and</u>

there exists a sequence $x_n \downarrow a$ $(y_n \uparrow a)$ *such that* $f(x_n)=0$ $(F(y_n)=0)$.

Proof. The sufficiency of conditions (i) and (ii) again follows easily from Young's criterion.

Before showing necessity we note that we can assume that $f(x) \geq 0$ since squaring preserves conditions (i) and (ii) and membership in $M(DB_1)$. It is also easy to show that if $f \in DB_1$ on I and $f(x) \neq 0$ for $x \in I$, then $1/f \in DB_1$ on I.

Since $g(x) \equiv 1 \in DB_1$, $f \in M(DB_1)$ implies $f \in DB_1$.

Case 1) Suppose f is discontinuous from the right at x=a and $f(x)>0$ on $(a,a+\delta]$. Choose $K>0$ such that there exists a sequence $p_n \downarrow a$ where $\lim_{n\to\infty} f(p_n)=K \neq f(a)$. Set

$$g(x) = \begin{cases} 1/f(a+\delta) & a+\delta \leq x \\ 1/f(x) & a<x<a+\delta \\ 1/K & x \leq a. \end{cases}$$

Then $g \in DB_1$, but $f(a)g(a) \neq 1$ and $f(x)g(x)=1$ on $(a,a+\delta]$.

Case 2) Suppose f is discontinuous from the right at x=a, $f(a)>0$ and that there exists $p_n \downarrow a$ such that $f(p_n)=0$. Set $E = \{x>a \mid f(x)<f(a)/2\}$. Since $f \in DB_1$, E is a bilaterally c-dense in itself F_σ set which is nonempty (because of the p_n) and contains no interval of the form $(a,a+\delta)$ since f has the Darboux property. For $x>a$ let $g \in DB_1$ such that $0<g(x) \leq 1$ for $x \in E$, $g(x)=0$ for $x \notin E$ and such that $\overline{\lim}_{x\to a^+} g(x)=1$, [1,Thm. 2.4]. For

$x \le a$ set $g(x)=1$. Then $g \in DB_1$ but $f(a)g(a)=f(a)$ and $f(x)g(x) \le f(a)/2$ for $a<x$. Thus fg does not have the Darboux property and the proof is complete.

References

1. A. M. Bruckner, Differentation of Real Functions, (preprint).

2. J. Young, A theorem in the theory of functions of a real variable, Rend. Circ. Mat. Palermo 24 (1907), 187-192.

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