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Decomposition of Approximate Derivatives

In [6], it was shown that:

If X is a closed subset of [0,1] and Y is any measure zero set, disjoint from X, which is also an F_σ and G_δ subset of [0,1], then there is an approximately differentiable function g such that

$$0 \le g(x) \le 1 \quad \text{for all } 0 \le x \le 1,$$
$$g(x) = 1 \quad \text{for all } x \text{ in } X,$$
$$g(x) = 0 \quad \text{for all } x \text{ in } Y.$$

What is of interest here is not the existence of such a function but the method by which it is obtained. A study of the construction shows that g is constructed by making a sequence of modifications, f_n, to a differentiable function f_o and that for each n, f_n is a differentiable function.

The purpose of the present paper is to show that, in a precise sense, all approximately differentiable functions can be obtained in this fashion. That is, each approximate derivative or approximately differentiable function can be considered as a "patchwork" of derivatives or differentiable functions. The "patchwork" is labeled the decomposition of the approximate derivatives.

This result will be obtained from the following new theorem relating the concepts of differentiability and approximate differentiability.

Theorem 1. Let Q be a measurable set and E a closed subset of the points of density of Q. Suppose $f:Q \to R$ is a measurable function possessing a finite approximate derivative at each point of E. Then E can be expressed as the countable union of closed sets E_n such that for each n and each x in E_n

$$E_n - \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'_{ap}(x).$$

Here the notation $E_n - \lim_{y \to x}$ means that we approach x only through the set $E_n \setminus \{x\}$. At an isolated point of E_n the conclusion is considered to hold vacuously. The sets E_n can be chosen to be perfect sets.

We note that Theorem 1 could be applied to get a version of the known result by Whitney [8]. However, we proceed to consider functions which are approximately differentiable everywhere in $[0,1]$.

Theorem 2. If $f: [0,1] \to R$ has a finite approximate derivative, f'_{ap}, at every point of $[0,1]$ then there is a sequence of perfect sets H_n and a sequence of differentiable functions h_n such that

i) $h_n(x) = f(x)$ ever H_n,

ii) $h'_n(x) = f'_{ap}(x)$ over H_n, and

iii) $\bigcup_{1 \leq n < \infty} H_n = [0,1]$.

The sequence (h_n, H_n) is called a decomposition of f. The corresponding sequence (h'_n, H_n) is the decomposition of the approximate derivative f'_{ap}.

It is obvious that the existence of the sets H_n presents a situation where the Baire category theorem can be usefully employed. We do so. As applications we present two corollaries. The first gives transparent proofs of two known theorems [1].

Corollary 1. Let $f: [0,1] \to R$ have a finite approximate derivative f'_{ap} everywhere in [0,1]. Then

a) There is a dense open set U such that f is differentiable on each component of U, and

b) The function f'_{ap} is Baire 1.

Basic to the concept of approximate differentiability is the idea that at a point x_o we may disregard the behavior of a function over certain "small" sets. Therefore it becomes natural to expect that knowledge of the behavior of f on a small set, such as nowhere dense sets of measure zero, would not permit the prediction of the values of f'_{ap} over this set. However, the next corollary shows that this is not quite true.

Corollary 2. Let $f: [0,1] \to R$ and $g: [0,1] \to R$ be two measurable functions. Suppose P is any perfect set such that $f(x) = g(x)$ over P. Suppose in addition that at every point of P f is approximately differentiable and g is differentiable. Then there is an open interval (a,b) with $(a,b) \cap P \neq \emptyset$ such that $f'_{ap} = g'$ at every point of $(a,b) \cap P$.

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