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A Denjoy Property for Distributions

The Denjoy property for functions is already familiar to readers of The Exchange. This is a report on the major results contained in the Ph.D. Thesis of Professor Charles R. Diminnie written under my direction in 1970 which dealt with the Denjoy property for derivatives of distributions. A distribution is a continuous linear functional defined on the space $C_c^\infty(R)$ of all infinitely differentiable functions having compact support where the topology makes $C_c^\infty(R)$ into a topological vector space and $\psi_n \to 0$ in the topology if there is a compact set $K$ containing the support of all of the functions $\psi_n$ and if $\{\psi_n^{(\kappa)}\}$ converges to $0$ uniformly, for all $\kappa = 0,1,2,\ldots$ (where, of course, $\psi^0 = \psi$). Many functions, including derivatives are distributions. In fact if $f$ is locally Perron integrable then a distribution, also denoted by $f$, can be defined by

$$\langle f, \psi \rangle = \int_R f(x)\psi(x)\,dx,$$

where integration is in the Perron sense. Diminnie's first main theorem deals with such distributions and the concept of convolution.

Definition. If $T$ is a distribution and $\psi \in C_c^\infty(R)$, then $T * \psi$ is the function defined by

$$T * \psi(x) = \langle T_t, \psi(x-t) \rangle$$

where the subscript $t$ on $T$ is used to indicate that the functional $T$ is operating on $\psi(x-t)$ considered as a function of $t$. It can easily be shown that $T * \psi$ is an infinitely differentiable function whose kth derivative is just $T * \psi^{(k)}$. By convolving with an approximation to the identity we obtain functions which approximate the distribution. Specifically, let $\varphi \in C_c^\infty(R)$ have the following properties:

1. $\varphi(x) \geq 0$ for all $x \in R$,
2. $\int_R \varphi(x)\,dx = 1$,
3. $\varphi(x) = 0$ if $|x| \geq 1$,
4. $\varphi(-x) = \varphi(x)$,

and 5. $\varphi'(x) > 0$ for $x \in (-1,0)$ and $\varphi'(x) < 0$ if $x \in (0,1)$

Moreover for $y > 0$ let $\varphi_y(t) = y\varphi(yt)$. Then it can be shown that for any distribution, $T$ $\{T * \varphi_y\}$, when considered as a family of distributions, converges to $T$ as $y$ tends to $\infty$, where $\lim_{y\to\infty} T_y = T$ means for each $\psi \in C_c^\infty(R)$

$$\lim_{y\to\infty} \langle T_y, \psi\rangle = \langle T, \psi\rangle.$$

More can be said if $T$ is a locally Perron integrable function.

<u>Theorem 1</u>. If $f$ is a locally Perron integrable function, then for a.e. $x$ $\lim_{y\to\infty} f * \varphi_y(x) = f(x)$.

One of the advantages distributions (or generalized functions as they are sometimes called) have over functions is that any distribution is (infinitely) differentiable.

<u>Definition</u>. Let $T$ be a distribution and let $\psi \in C_c^\infty(R)$. Then the derivative of $T$ is denoted by $DT$ and defined by

$$\langle DT, \psi\rangle = -\langle T, \psi'\rangle.$$

It can be easily shown using integration by parts that if $f$ is everywhere differentiable, then $Df$ is the distribution associated to the locally Perron integrable function $f'$. In a similar fashion it is possible to prove that for any distribution $T$ and any $\psi \in C_c^\infty(R)$,

$$DT * \psi = T * \psi'.$$

In order to define a Denjoy type property, Diminnie used the approximating functions $T * \varphi_y$, and introduced the following associated sets.

<u>Definition</u>. Let $T$ be a distribution and let $\alpha < \beta$. Then

$$E_{\alpha\beta}^{T} = \{(x,y) : \alpha < T * \varphi_y(x) < \beta, \quad y \geq 1\}$$

$$E_{\alpha\beta}^{T,N} = \{(x,y) : \alpha < T * \varphi_y(x) < \beta, \quad y \geq N\}.$$

One final definition is useful in stating the preliminary results.

<u>Definition</u>. Let $T$ be a distribution. Then $T \geq 0$ means $\langle T, \psi \rangle \geq 0$ for each $\psi \in C_c^\infty(R)$ such that $\psi(x) \geq 0$ for all $x \in R$.

<u>Theorem 2</u>. Let $T$ be a distribution. For all numbers $\alpha < \beta$ exactly one of the following occurs:

1. $m(E_{\alpha\beta}^{T}) > 0$
2. $T \leq \alpha$
3. $T \geq \beta$

($T \leq \alpha$ means $\alpha - T \geq 0$ where $\alpha$ is the distribution associated to the constant function, $\alpha$). Of course $m$ denotes two dimensional Lebesgue measure.

Theorem 3. Let $T$ be a distribution. For all real numbers $\alpha < \beta$ exactly one of the following occurs:

1. $E^T_{\alpha\beta}$ is empty
2. for all $N$ $m(E^{T,N}_{\alpha\beta}) > 0$, but $\lim_{N\to\infty} m(E^{T,N}_{\alpha\beta}) = 0$
3. for all $N$ $m(E^{T,N}_{\alpha\beta}) = \infty$.

Taking his hints from Theorem 3, Diminnie defined three classes of distributions:

Definition. A distribution $T$ belongs to class $0-S$ if for all $\alpha < \beta$ exactly one of the following occurs:

1. $T \leq \alpha$
2. $T \geq \beta$
3. there is a measurable set $E \subset R$ of positive measure and a number $N_0$ such that $E \times [N,\infty) \subseteq E^{T,N}_{\alpha\beta}$ for all $N \geq N_0$.

$T$ is in class $0-W$ if for all $\alpha < \beta$ exactly one of the following occurs:

1. $T \leq \alpha$
2. $T \geq \beta$
3. for each $N$, $m(E^{T,N}_{\alpha\beta}) = \infty$.

$T$ is in class $\theta$ for $\theta > 0$ if for all $\alpha < \beta$ exactly one of the following occurs:

1. $T \leq \alpha$
2. $T \geq \beta$
3. there are numbers $N_0$ and $K$ such that $m(E^{T,N}_{\alpha\beta}) \geq K(1/N)^{\theta}$.

Clearly class $0-S \subseteq$ class $0-W \in$ class $\theta$ for all $\theta > 0$, and class $\theta_1 \subseteq$ class $\theta_2$ whenever $0 < \theta_1 < \theta_2$. The first containment is proper. Diminnie showed that the distribution associated to the characteristic function of $(0,\infty)$ (the so-called Heaviside function) is not in class $0-S$ but is in class $0-W$. In order to prove that the third containment is proper he had to impose additional assumptions on the function $\varphi$ but with these was able to show in fact that $\bigcup_{\theta<\theta_0}$ class $\theta$ is a proper subset of class $\theta_0$ at least for $\theta_0 \leq 1$. Note that this implies that the second containment is proper.

The first theorem concerning these classes has as a corollary that the distribution associated to any derivative is in class $0-S$. But Diminnie's main results are contained in the next theorem.

Theorem 4. If $T = D^{(n)}g$ where $g$ is locally $L^\infty$, then $T$ is in class $n$ (class $0-W$ if $n=0$), and if $g$ is locally $L^p$ for $1 \leq p < \infty$ instead then $T$ is in class $(n+p^{-1})$. Finally if $T$ is the distribution associated to a (signed) measure, then $T$ belongs to class $(n+1)$.

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