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# On Generalized Derivatives

Terminology and introduction. Let $R = (-\infty, \infty)$. The words measure, almost etc. refer to the Lebesgue measure in R. If $S \subset R$ and $x \in R$, we write $d(x,S) = \inf\{|y-x|; y \in S\}$. If, moreover, S is measurable, then $m S$ denotes its measure. The notions of the kth Peano derivative $f_k$ and of the kth approximate Peano derivate $f_{(k)}$ of a function f are defined in the usual way (see, e.g., [1] and [3]); $f^{(k)}$ means the classical kth derivative.

Property Z of a real function g on R is defined as follows: If $x \in R$, $\varepsilon > 0$, $\eta > 0$, then there is a $\delta > 0$ such that for each interval $I \subset (x - \delta, x + \delta)$ with either $g(I) \subset [g(x), \infty)$ or $g(I) \subset (-\infty, g(x)]$ we have

$$(1) \qquad m\{y \in I;\ |g(y)-g(x)| \geqq \varepsilon\} \leqq \eta \cdot (m I + d(x,I)).$$

Property Z was introduced in [4] by Weil. He proved, among other things, that if $k > 0$ and if $f_k$ exists everywhere, then $f_k$ has Property Z. The proof, however, is complicated. In [1], Babcock generalized this result replacing $f_k$ by $f_{(k)}$, but a part of his proof (actually a part of the proof of Lemma 6.1) consists of hints how to modify the mentioned proof in [4]. The main purpose of this note is to prove a proposition (namely the present Theorem 1) enabling us to simplify the proof of Babcock's assertion which is stated here as Theorem 2. The present Theorem 3 is a simultaneous generalization of Lemma 3.4 in [2] and (with $j = k$) of Theorem 3 in [3].

At this opportunity I would like to express my thanks to Prof. C. E. Weil for his encouragement to write this note.

Lemma 1. Let $f$ be a monotone differentiable function on a bounded interval $I$. Let $\varepsilon > 0$, $\beta > 0$ and let $m\{x \in I ; |f'(x)| \geqq \varepsilon\} \geqq \beta$. Then there is an interval $J \subset I$ such that $mJ = \beta/4$ and that $|f| \geqq \varepsilon\beta/4$ on $J$.

Proof. We may suppose that $f' \geqq 0$ on $I$. Let $(a,b)$ be the interior of $I$. There is a $c \in [a,b]$ such that $f \leqq 0$ on $(a,c)$ and $f \geqq 0$ on $(c,b)$. Set $B = \{x \in I ; f'(x) \geqq \varepsilon\}$. If $m(B \cap (c,b)) \geqq \beta/2$ and if $x \in (b - \beta/4, b)$, then $f(x) \geqq \int_c^x f' \geqq$ $\geqq \varepsilon m(B \cap (c,x)) \geqq \varepsilon(m(B \cap (c,b)) - (b-x)) \geqq \varepsilon(\beta/2 - \beta/4) =$ $= \varepsilon\beta/4$. If $m(B \cap (a,c)) \geqq \beta/2$, then, analogously, $f \leqq -\varepsilon\beta/4$ on $(a, a+\beta/4)$.

Lemma 2. Let $I$ be a bounded interval and let $j$ be a natural number. Let $g$ be a function such that either $g^{(j)} \geqq 0$ on $I$ or $g^{(j)} \leqq 0$ on $I$. Let $\varepsilon > 0$, $\beta > 0$ and let $m\{x \in I ; |g^{(j)}(x)| \geqq \varepsilon\} \geqq \beta$. Then there is an interval $J \subset I$ such that $mJ = \beta/4^j$ and that $|g| \geqq \varepsilon\beta^j/4^{1+2+\cdots+j}$ on $J$.

(Follows by induction from Lemma 1.)

Theorem 1. Let $k$ be a natural number, let $x \in R$ and let $f$ be a function such that $f_{(k)}(x)$ exists. Define $P(y) = \sum_{i=0}^{k} (y-x)^i \cdot f_{(i)}(x)/i!$ $(y \in R)$. Let $\varepsilon > 0$, $\eta > 0$. Then there is a $\delta > 0$ with the following properties:

a) If $I$ is a subinterval of $(x-\delta, x+\delta)$, $j$ an integer with $0 < j \leqq k$ and if either $f^{(j)} \leqq P^{(j)}$ on $I$ or $f^{(j)} \geqq P^{(j)}$ on $I$, then

(2) $m\{y \in I ; |f^{(j)}(y) - P^{(j)}(y)| \geqq \varepsilon|y-x|^{k-j}\} \leqq \eta\cdot(mI + d(x,I))$.

b) If $I$ is any subinterval of $(x-\delta, x+\delta)$, then (2) holds with $j = 0$.

Proof. Let $g = f - P$, $\alpha = 4^{1+2+\cdots+k}$. There is a measurable set $A$ and a $\delta_1 > 0$ such that $x$ is a point of density of $A$ and that, for each $y \in A \cap (x-\delta_1, x+\delta_1)$, we have

$$(3) \qquad 3^k\alpha|g(y)| \leqq \varepsilon\eta^k|y-x|^k .$$

Further, there is a $\delta \in (0, \delta_1)$ such that, for each $h \in (0, 3\delta)$, we have

(4) $$3\cdot 4^j\, m([x-h,x+h]\setminus A) \leqq h\eta .$$

Now let $I$ be a subinterval of $(x-\delta,\ x+\delta)$ and let $j$ be an integer, $0 \leqq j \leqq k$. Let $B = \{y \in I\ ;\ |g^{(j)}(y)| \geqq \varepsilon|y-x|^{k-j}\}$, $\beta = \frac{1}{3}\, mB$, $h = mI + d(x,I)$. Now (2) becomes $3\beta \leqq \eta h$. Thus, we may suppose that $\beta > 0$. Let $C = B\setminus(x-\beta, x+\beta)$. Now $h < 3\delta$, $I \subset [x-h,x+h]$, $|g^{(j)}| \geqq \varepsilon\beta^{k-j}$ on $C$ and $mC \geqq \beta$. If $j > 0$ and if either $g^{(j)} \geqq 0$ on $I$ or $g^{(j)} \leqq 0$ on $I$, then, by Lemma 2, there is a set $S \subset I$ such that

(5) $$mS \geqq \beta/4^j$$

and that

(6) $$\alpha|g| \geqq \varepsilon\beta^{k-j}\cdot\beta^j = \varepsilon\beta^k \quad \text{on} \quad S\ ;$$

if $j = 0$, then these relations hold with $S = C$. If there is a $y \in S \cap A$, then, by (6) and (3), $3^k\varepsilon\beta^k \leqq \leqq 3^k\alpha|g(y)| \leqq \varepsilon\eta^k h^k$ so that $3\beta \leqq \eta h$. If $S \cap A = \emptyset$, then, by (5) and (4), $3\beta/4^j \leqq 3mS \leqq 3m([x-h,x+h]\setminus A) \leqq \leqq h\eta/4^j$ whence $3\beta \leqq \eta h$ again.

<u>Lemma 3</u>. Let $k$ be a natural number and let $f$ be a function such that $f_{(k)} \geqq 0$ on an interval $I$. Then $f^{(k)} = f_{(k)}$ on $I$.

(See [1], Theorem 4.1.)

Theorem 2. Let $k$ be a natural number and let $f$ be a function such that $f_{(k)}$ exists everywhere. Then $f_{(k)}$ has Property Z.

Proof. Let $x \in R$, $\varepsilon > 0$, $\eta > 0$. Choose a $\delta$ according to Theorem 1. If $P$ is as above, then, obviously, $P^{(k)} = f_{(k)}(x)$. Let $I$ be a subinterval of $(x-\delta, x+\delta)$ such that either $f_{(k)}(y) \leqq f_{(k)}(x)$ for each $y \in I$ or $f_{(k)}(y) \geqq f_{(k)}(x)$ for each $y \in I$. By Lemma 3, $f^{(k)} = f_{(k)}$ on $I$. Thus, (1) with $g = f_{(k)}$ is the same as (2) with $j = k$.

Lemma 4. Let $j$ be a natural number. Let $\varphi$ be a positive continuous function on an interval $I$. Let $g$ be a function such that $g_{(j)}$ exists (everywhere) on $I$ and let $|g_{(j)}| \geqq \varphi$ almost everywhere on $I$. Then $g^{(j)}$ exists on $I$ and either $g^{(j)} > 0$ on $I$ or $g^{(j)} < 0$ on $I$.

Proof. Let $x \in I$. There is an $\varepsilon > 0$ and an interval $J$ such that $x \in J \subset I$ and that $\varphi > \varepsilon$ on $J$. Thus, $|g_{(j)}| > \varepsilon$ almost everywhere on $J$. According to Corollary on p. 291 in [1] we have $|g_{(j)}| \geqq \varepsilon$ on $J$; in particular, $g_{(j)}(x) \neq 0$. It follows from Corollary on p. 290 in [1] that either $g_{(j)} > 0$ on $I$ or $g_{(j)} < 0$ on $I$. Now we apply Lemma 3.

Theorem 3. Let $j,k$ be integers, $0 \leqq j \leqq k$, $k > 0$. Let $x \in R$ and let $f$ be a function such that $f_{(k)}(x)$ exists. Define $P(y) = \sum_{i=0}^{k} (y-x)^i \cdot f_{(i)}(x)/i!$ $(y \in R)$. Let $\epsilon > 0$, $\eta > 0$. Then there is a $\delta > 0$ with the following property: If $L$ is a subinterval of $(x-\delta, x+\delta)$ such that $f_{(j)}$ exists on $L$ and that $|f_{(j)}(y) - P^{(j)}(y)| \geqq \epsilon |y-x|^{k-j}$ for almost all $y \in L$, then $mL \leqq \eta\, d(x,L)$.

Proof. Let $\delta$ be chosen according to Theorem 1, where $\eta$ is replaced by $\eta_1 = \eta/(1+\eta)$. Now let $L$ be as above. If $L \cap (x,\infty) \neq \emptyset$, set $I = L \cap (x,\infty)$; otherwise set $I = L \cap (-\infty, x)$. If $j > 0$, then it follows easily from Lemma 4 that either $f^{(j)} > P^{(j)}$ on $I$ or $f^{(j)} < P^{(j)}$ on $I$. According to Theorem 1 we have $mI \leqq \eta_1(mI + d(x,I))$ whence $mI \leqq \eta\, d(x,I)$. In particular, $d(x,I) > 0$ so that $I = L$.

## References


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