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BOUNDED VARIATION AND FOURIER SERIES

by

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The notion of bounded variation appeared quite early in the study of Fourier series in connection with the question of convergence and has continually reappeared in other connections. We shall see that the importance of functions of bounded variation in these connections derives from two sources:

(i) the oscillatory properties of the Dirichlet kernel,

(ii) the correspondence of these functions to signed Lebesgue-Stieltjes measures.

When we examine the relationship of bounded variation to (i) we will see that it is too primitive a notion in that results known for this class of functions admit of generalization to larger classes. On the other hand (ii) leads to results in which this notion seems to be intrinsic and these results cannot be extended to larger classes of functions. However, for these larger classes there may be alternate results of a similar structure.

We shall consider only real integrable functions of period 2π on the real line or R^m, $m > 1$, expanded as Fourier series with respect to the trigonometric system. Some of these results can be extended to other orthonormal systems, but the introduction of these questions would serve only to obscure our discussion.

1. Dirichlet-Jordan Theorem.

Dirichlet was probably the first to make a rigorous study of the convergence of Fourier series. The basic result associated with him is the following:

If f is continuous on $[0,2\pi]$ except for a finite number of points at which it has ordinary discontinuities and $[0,2\pi]$ is the union of finitely many intervals on each of which f is monotone, then the Fourier series of f converges to $f(x)$ at each point x of continuity and to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points of discontinuity.

Dirichlet actually extended these results to certain types of unbounded functions [1], but we shall not consider this here.

Clearly the class of functions considered by Dirichlet is a subclass of the functions of bounded variation (BV) and, as is well known, the result may be extended to them. In addition to the pointwise convergence, we also have that the Fourier series converges uniformly to f on any closed interval of points of continuity.

Let us consider a proof of this extended result, usually referred to as the Dirichlet-Jordan theorem, a proof which is not only quite different from that usually given, but which reveals more about the nature of the result than the usual argument and admits of generalization.

Let f be a real integrable function of period 2π with Fourier series

$$S(f) = \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt.$$

The partial sums of this series are given by

$$S_n(x) = S_n(x,f) = \frac{a_0}{2} + \sum_1^n (a_k \cos kx + b_k \sin kx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,dt$$

where the Dirichlet kernel

$$D_n(t) = \frac{1}{2} + \sum_1^n \cos kt = \sin(n+\tfrac{1}{2})t/2\sin\tfrac{1}{2}t.$$

It is easy to see that for any $\delta > 0$

$$S_n(x) - f(x) = \frac{1}{\pi}\int_0^\delta (f(x+t)+f(x-t)-2f(x))\frac{\sin nt}{t}\,dt + o(1)$$

for each x and uniformly on intervals in which f is bounded [2,I,p.55].

We shall assume that for each x, $f(x) = \frac{1}{2}(f(x+0)+f(x-0))$ and, therefore, the integral above is the sum of

$$\int_0^\delta (f(x+t)-f(x+0))\frac{\sin nt}{t}\,dt \quad \text{and} \quad \int_0^\delta (f(x-t)-f(x-0))\frac{\sin nt}{t}\,dt.$$

We will confine our attention to the first of these. The other may be treated in an analogous manner. Letting $h(t)=h_x(t)=f(x+t)-f(x+0)$, we see that

$$\left|\int_0^{\pi/n} h(t)\frac{\sin nt}{t}\,dt\right| \le \pi \sup_{0<t<\pi/n} |h(t)| = o(1)$$

for each x and uniformly on closed intervals of points of continuity. This expression is also uniformly bounded if f is bounded. Then

$$\int_{\pi/n}^{\delta} h(t)\frac{\sin nt}{t}\,dt = \sum_1^N \int_{k\pi/n}^{(k+1)\pi/n} h(t)\frac{\sin nt}{t}\,dt + \int_{(N+1)\pi/n}^{\delta} \cdots$$

$$= I_1 + I_2$$

where $N+1=[\frac{n\delta}{\pi}]$. Clearly $I_2=o(1)$ uniformly in x as $n\to\infty$ on intervals in which f is bounded. Consider the sum

$$I_1=\sum_1^N \int_0^{\pi/n} h(t+k\pi/n)(-1)^k \frac{\sin nt}{t+k\pi/n}\,dt$$

$$= \int_0^{\pi} \sum_1^N h\left(\frac{t+k\pi}{n}\right)(-1)^k \frac{\sin t}{t+k\pi}\,dt\,.$$

We shall show that, under appropriate conditions on f, this last integrand is bounded, $o(1)$ for each x, and uniformly $o(1)$ on closed intervals of points of continuity of f.

If N is even, the integrand may be written as

$$-\sum_{k=1}^{N-1}{}^{*} \left[h\left(\frac{t+k\pi}{n}\right)\frac{1}{t+k\pi} - h\left(\frac{t+(k+1)\pi}{n}\right)\frac{1}{t+(k+1)\pi}\right],$$

suppressing $\sin t$, since it is bounded and of constant sign.

Here * indicates that k takes only odd values. If N is odd, then

$$\int_{N\pi/n}^{(N+1)\pi/n} h(t)\,\frac{\sin nt}{t}\,dt = o(1)$$

just as I_2 did, and, by removing this term, we can reduce the problem to one in which the sum has an even number of terms. Thus we may assume N to be even without loss of generality.

The general term of the sum under consideration equals

$$[h(\frac{t+k\pi}{n}) - h(\frac{t+(k+1)\pi}{n})]\,\frac{1}{t+k\pi} + h(\frac{t+(k+1)\pi}{n})\,[\frac{1}{t+k\pi} - \frac{1}{t+(k+1)\pi}] \,.$$

Given $\varepsilon > 0$ and choosing N_0 so that $\sum_{N_0+1}^{\infty} {}^{*}\, 1/k^2 < \varepsilon$, we have

$$\Big|\sum_{1}^{N-1} {}^{*}\, h(\frac{t+(k+1)\pi}{n})\,[\frac{1}{t+k\pi} - \frac{1}{t+(k+1)\pi}]\Big|$$

$$\leq \sum_{1}^{N-1} {}^{*}\, |f(x+\frac{t+(k+1)\pi}{n}) - f(x+0)|/k^2 = \sum_{1}^{N_0} {}^{*} + \sum_{N_0+1}^{N-1} {}^{*}$$

and the second sum is bounded by $2\varepsilon \sup_{[0,2\pi]} |f(x)|$. The first sum is bounded by

$$\sup_{0 < u \leq (N_0+2)\pi/n} |f(x+u)-f(x+0)| \cdot \sum_{1}^{N_0} {}^{*}\, 1/k^2 = o(1)$$

as $n \to \infty$ for each x, is $o(1)$ uniformly for x in a closed interval of points of continuity, and its integral is bounded uniformly in n and x on any interval of boundedness of f .

The most interesting part of this argument is the consideration of the remaining portion of the integrand. Now

$$(*)\qquad \left| \sum_{k=1}^{N-1}{}^{*}\left[h\left(\frac{t+k\pi}{n}\right) - h\left(\frac{t+(k+1)\pi}{n}\right)\right] \frac{1}{t+k\pi}\right|$$

$$\leq \sum_{k=1}^{N-1}{}^{*}\left|h\left(\frac{t+k\pi}{n}\right) - h\left(\frac{t+(k+1)\pi}{n}\right)\right| \frac{1}{k}.$$

Let $\overline{f}(t) = \begin{cases} f(t) & \text{on } (x,x+\delta] \\ f(x+0), & t=x \end{cases}$. If we now assume that $f \in BV$ we see that the sum (*) is dominated by the variation of $\overline{f}$ on $[x,x+\delta]$ and this may be made less than ε by choosing δ sufficiently small, since $\overline{f}$ is continuous on the right at x. If f is continuous at every point of a closed interval I, the variation of f in $[x,x+\delta]$, $x \in I$, is $o(1)$ uniformly on I as $\delta \to 0$.

This argument establishes the Dirichlet-Jordan theorem, but is clear from the form of (*) that the hypothesis $f \in BV$ is much too strong.

A function f on an interval I is said to be of harmonic bounded variation (HBV) if

$$\sum_{1}^{\infty} |f(a_n)-f(b_n)|/n < \infty$$

for every sequence of non-overlapping intervals (a_n,b_n) in I. This is equivalent to requiring that such sums over finite collections of intervals be uniformly bounded. The supremum of such sums is called the (total) harmonic variation of f on I.

We observe that the sum (*) is bounded by the harmonic variation of $\overline{f}$ on $[x, x+\delta]$ and that the remarks made for the ordinary variation over this interval apply with equal force to the harmonic variation.

This proof that a theorem of Dirichlet-Jordan type holds for functions of class HBV is new. The original argument [3] showed somewhat more in one sense, that functions in HBV satisfy the Lebesgue convergence test, but it did not show as clearly how intimately this notion is related to the structure of the Dirichlet kernel, nor did it show, as this argument does, that $\{S_n(f)\}$ is <u>uniformly</u> <u>bounded</u> for $f \in \mathrm{HBV}$.

By considering sums of the form

$$\Sigma |f(a_b)-f(b_n)|/\lambda_n$$

where $\Lambda = \{\lambda_n\}$ is a nondecreasing sequence of real numbers with $\Sigma\, 1/\lambda_n = \infty$, we may define the class of functions of Λ-bounded variation (ΛBV). For a full discussion of the properties of functions of HBV and ΛBV see our papers [3,4,5].

Suppose $\Phi \geq 0$ is a convex function, $\Phi(x) = o(x)$ as $x \to 0+$, $\Phi(x)/x \to \infty$ as $x \to \infty$ and $\Phi(0) = 0$. Let $\Psi(x) = \sup\{xy-\Phi(y) \mid y \geq 0\}$. Then we have

$$xy \leq \Phi(x) + \Psi(y),$$

which is Young's inequality.

The notion of Φ-bounded variation (Φ-BV) was introduced by L. C. Young [6]. Salem showed that the Fourier series of functions of Φ-bounded variation converge uniformly if $\Sigma\,\Psi(1/n) < \infty$ [7]. L. C. Young had shown this for various particular functions Φ and Wiener [8] had done this for $\Phi(x) = x^p$, $p > 1$.

We could easily show that Salem's conditions can be applied to the sum (*) to obtain a theorem of Dirichlet-Jordan type. However it is obvious that $\Phi\text{-BV} \subset \text{HBV}$ if $\Sigma\,\Psi(1/n) < \infty$ since

$$\Sigma\,|f(a_n)-f(b_n)|/n \le \Sigma\,\Phi(|f(a_n)-f(b_n)|) + \Sigma\,\Psi(1/n)$$

by Young's inequality.

Another form of generalized bounded variation was considered by Garsia and Sawyer [9]. The Banach indicatrix $n(y)$ of a continuous function $f(x)$ is defined to be the cardinality of $f^{-1}(y)$ if this set is finite and $+\infty$ otherwise. It is classical that the total variation of f is $\int n(y)dy$. Garsia and Sawyer showed that the Fourier series of functions for which $\int \log^+ n(y)dy < \infty$ converge uniformly. Here $\log^+(x)$ denotes the positive part of $\log x$.

The notion of Banach indicatrix can be generalized to regulated functions, i.e., those with right and left limits at each point and it can be shown that a theorem of Dirichlet-Jordan type holds for the functions with $\int \log^+ n(y)dy < \infty$ [3]. This is best shown by observing that these functions are in HBV, which is a consequence of the following lemma of Goffman [10]:

LEMMA. Let $\{E_i\}$ be a sequence of μ-measurable sets of a measure space $(\mathcal{X},\mathcal{A},\mu)$ and let $S_\infty = \overline{\lim} E_i$ and S_n, $n = 1,2,3,\ldots$, be the set of points belonging to exactly n of the sets E_i. If $\{a_i\}$ is a decreasing sequence of non-negative real numbers, then

$$\sum_1^\infty a_i \mu(E_i) \le \sum_1^\infty \mu(S_n)\left(\sum_1^n a_i\right) + \sum_1^\infty a_i \mu(S_\infty) .$$

From this lemma one obtains the following immediate corollary by selecting $a_i = 1/i$:

COROLLARY. If $\mu(\mathcal{X}) < \infty$, then $\sum_1^\infty \mu(E_i)/i = \infty$ implies that $\mu(S_\infty) \neq 0$ or $\sum_1^\infty \mu(S_n) \log n = \infty$.

Given a regulated function f, adjoin to its graph the line segment connecting $(x, \underline{\lim}_{t\to x} f(t))$ and $(x, \overline{\lim}_{t\to x} f(t))$ for each point x at which f is discontinuous, assuming, for simplicity, that at such points $f(x)$ is between $f(x+0)$ and $f(x-0)$. Let G denote the union of the graph and these segments. Let $n(y)$ be the cardinality of $\{x \mid (x,y) \in G\}$ if this is finite and $+\infty$ otherwise. It is easy to see that if f is defined on $[a,b]$, $a \le x_0 < x_1 = b$ and $((x_0,y_0)$ and (x_1,y_1) are in G, then for each y between y_0 and y_1 there is an $x \in [x_0,x_1]$ such that $(x,y) \in G$.

If $f \notin \mathrm{HBV}$, but is regulated on $[a,b]$, there is a sequence of non-overlapping intervals $[a_n,b_n]$ such that $\sum_{1}^{\infty} |f(a_n)-f(b_n)|/n = \infty$. Let E_n be the interval with endpoints $f(a_n)$ and $f(b_n)$. If $y \in S_k$, then $n(y) \geq k$. Thus if $I = [\inf f, \sup f]$, then

$$\int_I \log n(y)dy \geq \sum_{1}^{\infty} m(S_n)\log n + \sum \frac{1}{n} m(S_\infty) = \infty ,$$

since $\sum m(E_n)/n = \infty$ and $m(I) < \infty$. Hence f does not satisfy the Garsia-Sawyer condition.

We have now seen that the class of functions of bounded variation in the Dirichlet-Jordan theorem can be replaced by the classes HBV, Φ-BV (with $\sum \Psi(1/n) < \infty$), and functions for which $\int \log^+ n(y)dy < \infty$. We have also seen that HBV contains the other classes and that its definition is closely related to the oscillatory nature and the magnitude of the Dirichlet kernel. It may also be shown that a theorem of Dirichlet-Jordan type does <u>not</u> hold for ΛBV if ΛBV is not a subclass of HBV [3].

2. Magnitude of the Fourier coefficients.

If $f \geq 0$ is non-decreasing, we have, for some $\theta \in (0,2\pi)$

$$\pi a_k = \int_0^{2\pi} f(t)\cos ktdt = f(2\pi)\int_\theta^{2\pi} \cos ktdt$$

by the second mean-value theorem. Thus

$$|a_k| \leq 2f(2\pi)/\pi k$$

and, writing $f = p-n$, where p and n are the positive and negative variations of f, we see that $|a_k|$ and, in much the same way, $|b_k|$ do not exceed

$$2(p(2\pi)+n(2\pi))/\pi k = \frac{2}{\pi k} V .$$

Here V denotes the total variation of f on $[0,2\pi]$.

Alternatively, one may consider the Fourier-Stieltjes series $S(df)$ whose coefficients are given by

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \cos ktdf(t) \quad \text{and} \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \sin ktdf(t) .$$

The result above is seen to be equivalent to the statement: the coefficients of $S(df)$ are bounded.

Since $\sum_1^{\infty} \sin kx/k$ is the Fourier series of a function of bounded variation, we see that this estimate cannot be improved. Even if the function is required to be continuous, it may be shown that the coefficients need not be $o(1/n)$ [2,I,p.196].

It is interesting to note that this estimate leads to another proof of the Dirichlet-Jordan theorem. A theorem of Hardy asserts that if $\sum_1^{\infty} u_n$ is $(C,1)$ summable (the method of the first arithmetic mean) and $u_n = O(1/n)$, then $\sum_1^{\infty} u_n$ converges. Fejér's theorem asserts that if $f \in L^1$ then it is $(C,1)$ summable to $\frac{1}{2}[f(x+0)+f(x-0)]$ at each point where $f(x\pm 0)$ exist and uniformly summable on closed intervals of points of continuity. Our estimate of a_k and b_k with these results yields the Dirichlet-Jordan theorem.

We have seen that the estimation of the magnitude of the coefficients of $S(f)$ for $f \in BV$ is an immediate consequence of the Jordan decomposition. Since this decomposition is the distinguishing feature of the class BV, the argument affords no insight into the question of such estimates for other classes. However for one class of functions of generalized variation an estimate is available from another source.

Consider the class ΛBV with $\Lambda = \{n^{\beta+1}\}$, $-1 \le \beta \le 0$. We have shown that the Cesàro means of $S(f)$ of order β are bounded if $f \in \{n^{\beta+1}\} - BV$ [4]. It is known that if the means are bounded, then the general term of the series is $O(n^{\beta})$.

The result may be improved if a certain continuity condition is imposed. Let $f \in \Lambda BV$ and let $\Lambda_m = \{\lambda_{n+m}\}$, $m = 1,2,\ldots$. We say that f is <u>continuous in Λ-variation</u> if the Λ_m-variation of f tends to 0 as $m \to \infty$. If we assume that f is continuous in $\{n^{\beta+1}\}$-variation, $-1 < \beta < 0$, then $S(f)$ is Cesàro summable of order β, which implies that the Fourier coefficients are $o(n^{\beta})$.

If we assume $f \in \Lambda BV$, then by a method independent of summability considerations we can obtain $O(\lambda_n/n)$ as an estimate of the Fourier coefficients, which includes the estimates given for BV and $\{n^{\beta+1}\}$-BV. We will consider b_n. We have

$$\pi b_n = \int_{-\pi}^{\pi} f(t)\sin nt\,dt = \int_0^{\pi} \ldots + \int_{-\pi}^{0} \ldots .$$

We will estimate only the first integral; the second may be treated analogously. Now

$$\int_0^\pi f(t)\sin ntdt = \Sigma(-1)^k \int_0^{\pi/n} f(t+k\pi/n)\sin ntdt$$

$$= \frac{1}{n}\Sigma(-1)^k \int_0^\pi f(\frac{t+k\pi}{n})\sin tdt .$$

If this sum has an even number of terms it may be written as

$$\frac{1}{n}\int_0^\pi \Sigma^*[f(\frac{t+(k-1)\pi}{n}) - f(\frac{t+k\pi}{n})]\sin tdt ,$$

where * again denotes summation over odd indices. If the sum has an odd number of terms, then the integral of the term with largest k tends to zero as $n\to\infty$ and, without loss of generality, we may consider the sum with this term deleted. Now

$$\frac{1}{n}|\int_0^\pi[\Sigma^*\cdots]\sin ntdt| \le \frac{1}{n}\int_0^\pi \Sigma^*|f(\frac{t+(k-1)\pi}{n}) - f(\frac{t+k\pi}{n})|dt$$

$$\le \frac{1}{n}\int \Sigma^* \frac{1}{\lambda_k}|\cdots|\lambda_k dt .$$

Applying Abel's transformation, we see that this expression is $O(\lambda_n/n)$.

It is natural to ask if it is possible to determine that a function f is of bounded variation from the magnitude of its Fourier coefficients. In this connection we have the following theorem of Lorentz [11, p.211]:

A function f is of bounded variation if

(i) for $1\le p\le 2$, $(\sum_n^\infty |a_k|^p+|b_k|^p)^{1/p} = O(1/n)$

or

(ii) for $2\le p\le\infty$, $(\Sigma |a_k|^p+|b_k|^p)^{1/p} = O(\frac{1}{n^{1/2+1/q}})$,
where $1/p+1/q=1$.

We do not know of any such results for functions of generalized bounded variation.

3. Absolute Convergence

A theorem of Denjoy and Lusin [2,I,p.232] asserts that if $S(f)$ converges absolutely on a set of positive measure, then $\Sigma|a_k|+|b_k|$ converges. The convergence of this series implies that $S(f)$ converges absolutely everywhere as well as uniformly.

We see then that if we seek a condition which implies absolute convergence, it must imply continuity. It is natural, therefore, that such conditions generally include restrictions on $\omega(\delta)=\omega_f(\delta)$, the modulus of continuity of f. For example, we have the following theorem of Zygmund [2,I,p.241]:

If f is of bounded variation and $\omega(\delta)=O(\delta^{\alpha})$ for some $\alpha>0$, then $S(f)$ converges absolutely.

Zygmund notes that the condition on ω is not superfluous, for even if f is absolutely continuous, $S(f)$ need not converge absolutely.

Hirschman [12, lemma D] has shown that BV in Zygmund's theorem can be replaced by bounded Wiener p-variation if $1\leq p<2$.

For ΛBV we have shown the following [3]:

If $f\in\Lambda BV$, then $S(f)$ converges uniformly if $\Sigma\lambda_n^{1/2}n^{-1}\omega^{1/2}(\pi/n)$ is a convergent monotone series.

If $f \in HBV$, then $S(f)$ converges absolutely if $\sum n^{-1/2} \omega^{1/2}(\pi/n)$ converges.

Note that $\Lambda BV = BV$ if $\lambda_n \equiv 1$, and if $\omega(\delta) = O(\delta^{\alpha})$, $\alpha > 0$, then $\lambda_n \equiv 1$ implies

$$\lambda_n^{1/2} n^{-1} \omega^{1/2}(\pi/n) = O(n^{-(1+\alpha)})$$

and so Zygmund's theorem is a consequence of ours.

An interesting analysis of the connections between these results and related results using the integrated modulus of continuity has been done by Wik [13]. A discussion of various related counterexamples is to be found in a paper of Onneweer [14].

4. Continuity

We have mentioned that, although the Fourier coefficients of a function of bounded variation are $O(1/n)$, requiring that the function is also continuous will not imply that the coefficients are $o(1/n)$. It is possible, however, to give conditions on the coefficients which imply the continuity of the function.

Let $\rho_k^2 = a_k^2 + b_k^2$. Consider the conditions:

(I) $\sum_1^n k^2 \rho_k^2 = o(n)$

(II) $\sum_1^n k \rho_k = o(n)$

(III) $\sum_1^n \rho_k = o(\log n)$

(IV) $\sum_1^n \rho_k^2 = o(1/n)$.

Let W denote the class of regulated functions without external saltus, i.e., $f \in W$ if and only if $f(x+0)$ and $f(x-0)$ exist and

$$\min(f(x+0), f(x-0)) \leq f(x) \leq \max(f(x+0), f(x-0))$$

for each x. Let V_p denote the class of functions of bounded Wiener p-variation.

Wiener has shown that (II) is a necessary and sufficient condition for functions in $W \cap BV$ to be continuous [8].

Golubov has shown that for functions in $W \cap V_p$, $1 \leq p < 2$, each of the conditions (I)-(IV) is necessary and sufficient for continuity, while for functions in W or functions in $W \cap V_p$ with $p \geq 2$, these conditions are sufficient for continuity, but <u>no</u> condition on ρ_k can be necessary [15]. He also observed that these conditions have the following relation, (IV) => (I) => (II) => (III).

E. Cohen has generalized the results of Wiener and Golubov to functions of Φ-BV [16]. The class of functions which she considers is somewhat different from ours. She makes the additional assumption that Φ is strictly increasing. V_Φ is the class of functions of Φ-bounded variation and V_Φ^* is the class of functions f such that $kf \in V_\Phi$ for some $k \neq 0$.

She has shown that if $x^2 = o(\Phi(x))$ as $x \to 0$, then each of the conditions (I)-(IV) is necessary and sufficient for the functions of $W \cap V_\Phi^*$ to be continuous, but if $\underline{\lim}\, u^2/\Phi(u) \neq 0$, then <u>no</u> condition on ρ_k can be necessary.

5. Approximating functions by $S_n(f)$

The theorem of Salem on Φ-BV may be viewed as a result on the degree of approximation of a function $f \in C$, the continuous functions, by the partial sums, $S_n(f)$, of its Fourier series:

If $f \in C \cap \Phi$-BV and $\Sigma\, \Psi(1/n) < \infty$, then

$$\|f-S_n(f)\|_C = o(1) .$$

E. Cohen [16] has extended this to functions whose r-th fractional derivatives are in $C \cap V_\Phi^*$, with $\Sigma\, \Psi(1/n) < \infty$ and $0 \le r < \infty$. She has shown that

$$\|f-S_n(f)\|_C = o(n^{-r}) .$$

She has found such estimates for functions with discontinuous r-th derivative and also obtained estimates in Orlicz space norms.

This work generalizes results of this type of Golubov [17], who estimated the L^q norm of $f-S_n(f)$ when the r-th derivative of f is in V_p and $q > p$.

6. Parseval's formula.

If f and g are in L^2, the classical Parseval formula is

$$\frac{1}{\pi}\int_0^{2\pi} fg\,dx = \frac{1}{2}a_0 a_0' + \sum_1^{\infty}(a_k a_k' + b_k b_k') ,$$

the a_k' and b_k' being the Fourier coefficients of g. Two classes of functions are called <u>complementary</u> if, whenever f is in one and g is in the other, the series above is summable by some method.

Pairs of complementary classes are, for example, (L^p, L^q) with $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, $(L\log L, L_\infty)$ and (L, BV). In each of these examples the series converges [2, I, pp. 159 & 267].

We shall show that (L, HBV) is a pair of complementary classes and that the series above converges.

Let δ_n denote the difference between the integral and the n-th partial sum of the series above. Then

$$|\delta_n| = \frac{1}{2\pi}\left|\int_0^{2\pi}(g-S_n(g))f\,dx\right| \le \frac{1}{2\pi}\int_0^{2\pi}|g-S_n(g)|\,|f|\,dx .$$

In §1 we showed that $S_n(g)$ not only converges to g, but that it is uniformly bounded. Hence $|g-S_n(g)|\,|f|$ is bounded by an integrable function and tends to zero almost everywhere. Thus $\delta_n \to 0$.

We know no results for classes ΛBV which are not contained in HBV. Since a function $g \in \Lambda BV$ is bounded, $fg \in L$ for $f \in L$. It appears to be a reasonable conjecture that $(L, \Lambda BV)$ is a complementary class with an appropriate regular summability method which depends on Λ.

7. The F. and M. Riesz theorem

The conjugate Fourier series of f is

$$\tilde{S}(f) = \Sigma(a_k \sin kx - b_k \cos kx) .$$

The summation problem for this series leads to the expression

$$-\frac{2}{\pi}\int_{\epsilon}^{\pi} \frac{f(x+t)-f(x-t)}{2\tan t/2}\, dt .$$

The limit of this as $\epsilon \to +0$ is denoted by $\tilde{f}(x)$ and the function $\tilde{f}$ is said to be conjugate to f. It is well known that for $f \in L$, $\tilde{f}(x)$ exists almost everywhere.

The Abel mean of a series Σu_k is $\Sigma u_k r^k$, $0 \le r < 1$. Denoting the Abel means of $S(f)$ and $\tilde{S}(f)$ by $S(f,r)$ and $\tilde{S}(f,r)$ and letting $z = re^{ix}$, we see that

$$S(f,r) + i\tilde{S}(f,r) = \frac{1}{2}a_0 + \Sigma(a_k - ib_k)z^k$$

is a function analytic in the unit circle.

A necessary and sufficient condition for the convergence of $\tilde{S}(f)$ at a point is the existence of $\tilde{f}$. It is also known that if $f(x+0)-f(x-0) \neq 0$, then $\tilde{S}(f)$ diverges at x. Thus if f is regulated and $\tilde{S}(f)$ converges everywhere, we see that f is continuous. If we assume that $\tilde{f}$ is regulated and $S(f)$ converges everywhere, we have that $\tilde{f}$ is continuous.

Suppose now that $F(z) = \Sigma c_n z^n$ with $z = re^{ix}$ is a power series converging for $|z| < 1$ and its real and imaginary parts for $r = 1$ are Fourier series of functions in HBV. Then $F(e^{ix}) = \lim_{r \to 1-0} F(re^{ix})$ is continuous.

The theorem of F. and M. Riesz asserts that if the real and imaginary parts of F for $r = 1$ are Fourier series of functions in BV, then $F(e^{ix})$ is <u>absolutely</u> continuous.

The proof of this result is rather deep and would lead us too far afield. For interesting discussions of this result and its ramifications see [2, I, pp. 285-288] and [18, pp. 88-91].

The result given above for HBV is much less deep than the F. and M. Riesz theorem, which leads us to conjecture that it may be refined, that the HBV hypothesis may imply that $F(e^{ix})$ satisfies a generalized absolute continuity condition.

8. Multiple Fourier Series

The Riemann localization principle for functions of one variable asserts that if $f \in L$ and $f(x) \equiv 0$ for $x \in (a,b)$, then $S_n(f) \to 0$ uniformly on compact subsets of (a,b). For functions of several variables, no such result can be obtained without additional assumptions. If $S_n = S_{n_1,\ldots,n_m}$ is the n-th partial sum of $S(f)$, the Fourier series of an integrable function on $[-\pi,\pi]^m$, $m > 1$, and convergence means existence of $\lim S_n$ as

$\min n_i \to \infty$, then even for a continuous function, localization may fail.

There are various types of additional assumptions one might make, for example,

1. require $f = 0$ on a larger set,
2. make additional global requirements on f,
3. replace convergence by other limiting procedures,
4. replace rectangular partial sums by other partial sums,

and various combinations of these [2, II, Chap. 17].

Tonelli [19, Chap. 9] introduced a notion of bounded variation which yielded a pointwise convergence theorem for functions of two variables. This implies a pointwise localization principle: $f \equiv 0$ on G open implies $S_n(f) \to 0$ at each point of G. The usual localization was obtained only with additional (but unnecessary) hypotheses. Cesari [20] improved on Tonelli's results with his introduction of generalized bounded variation which guarantees localization and almost everywhere convergence and which has had many other applications. This notion of Cesari may be expressed as follows:

f is measurable and, corresponding to each coordinate direction, there is an equivalent function which is of bounded variation on almost every line in that direction, and whose total variation on those lines is an integrable function of the remaining (m-1) variables.

With Goffman, we have recently extended these results by replacing bounded variation on the lines in the coordinate directions by ΛBV [21,22]. We have shown that, in R^2, replacing BV by HBV yields the localization principle for convergence of rectangular partial sums. We have also shown that if ΛBV is not contained in HBV, then ΛBV contains a function for which the localization principle fails even for square partial sums (S_{n_1,n_2} with $n_1 = n_2$).

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