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Generalizations of L'Hospital's Rule

The following is a version of the well-known l'Hôpital rule.

Theorem 1. Let F, G be two real-valued functions defined on the open interval (a,b). Suppose that the following conditions are satisfied:

(H1) F, G are differentiable on (a,b) and $G'(x) \neq 0$ for all x in (a,b);

(H2) $\lim_{x\to a+} [F'(x)/G'(x)] = A$;

(H3) $\lim_{x\to a+} F(x) = 0 = \lim_{x\to a+} G(x)$.

Then

(C) $\lim_{x\to a+} [F(x)/G(x)] = A$.

It is known (cf. [4]) that the existence of the ordinary limit in (H2) cannot be weakened to that of the approximate limit even if one only wants to conclude that $\text{aplim}_{x\to a+} [F(x)/G(x)] = A$. Sandwiched in between the concept of the ordinary limit and that of the approximate limit, a concept called the essential limit is introduced, and using it we show that not only the condition (H2) but also the conditions (H1) and (H3) can be weakened so that the conclusion (C) still holds true. In fact, we have the following results:

Theorem I. Let a,b be two extended real numbers with $-\infty \leq a < b \leq +\infty$, and let F,G be real-valued functions defined on the open interval (a,b). Suppose that the following conditions are satisfied:

(h1) the approximate derivatives $F_{(1)}(x)$ and $G_{(1)}(x)$ exist finitely and $G_{(1)}(x) > 0$ for almost all x in (a,b);

(h2) $\text{esslim}_{x\to a+} [F_{(1)}(x)/G_{(1)}(x)] = A$;

(h3) (i) $\lim_{x\to a+} F(x) = 0 = \lim_{x\to a+} G(x)$, or

(ii) $\lim_{x\to a+} G(x) = -\infty$;

(h4) G, pG-F and F-qG are [ℓACG] and have the Darboux property on every compact subinterval of (a,b) for all real numbers p in an open interval (A, p_o) provided that $A \neq +\infty$, and for all real numbers q in an open interval (q_o, A) provided that $A \neq -\infty$.

Then

(C) $\lim_{x\to a+} [F(x)/G(x)] = A$.

Theorem $II_n (n \geq 2)$. Let a,b be two extended real numbers with $-\infty \leq a < b \leq +\infty$, and let F,G be real-valued functions defined on the open interval (a,b) such that the $(n-1)^{st}$ approximate Peano derivatives $F_{(n-1)}(x)$ and $G_{(n-1)}(x)$ exist finitely for all x in (a,b). Suppose that the following conditions are satisfied:

(hl.n) the n^{th} approximate Peano derivatives $F_{(n)}(x)$ and $G_{(n)}(x)$ exist finitely and $G_{(n)}(x) > 0$ for almost all x in (a,b);

(h2.n) $\text{esslim}_{x\to a+} [F_{(n)}(x)/G_{(n)}(x)] = A$;

(h3.n) (i) $\lim_{x\to a+} F_{(n-1)}(x) = 0 = \lim_{x\to a+} G_{(n-1)}(x)$, or

(ii) $\lim_{x\to a+} G_{(n-1)}(x) = -\infty$;

(h4.n) $u_o G_{(n)}(x) > -\infty$, $u_o (pG-F)_{(n)}(x) > -\infty$ and $u_o (F-qG)_{(n)}(x) > -\infty$ for nearly all x in (a,b) and for all p in an open interval (A, p_o) provided that $A \neq +\infty$ and for all q in an open interval (q_o, A) provided that $A \neq -\infty$.

Then

$$(Cn) \quad \lim_{x\to a+} [F_{(n-1)}(x)/G_{(n-1)}(x)] = A.$$

Proofs of the above theorems to appear in [3], model that of the ordinary l'Hopital rule given by Rudin in [5], except that instead of using a generalized mean value theorem we have to appeal directly to some monotonicity theorems in [1] and [2]. We also note that some further remarks concerning the theorems are also included in [3].

References:

[1] C.-M. Lee, On the approximate Peano derivatives, J. London Math. Soc. (2) 12 (1976), 475-478.

[2] ________, An analogue of the theorem of Hake-Alexandroff-Looman, to appear in Fund. Math.

[3] ________, Generalizations of l'Hôpital's rule, to appear in Proc. Amer. Math. Soc.

[4] ________ and R. J. O'Malley, The second approximate derivatives and the second approximate Peano derivatives, Bull. Inst. Math., Acad. Sinica, Taiwan, 3 (1975), 193-197.

[5] W. Rudin, Principles of Math. Analysis, McGraw-Hill, New York (1953).

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