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On Cluster Sets And Essential Cluster Sets

Let H denote the open upper half plane above the real line R. Let x denote points on R. For fixed θ, $0<\theta<\pi$, let $L_\theta(x)$ be the half ray in H in the direction of θ with one end point at x. Also let $\sigma \subset H$ denote a sector with vertex at the origin and let $\sigma(x)$ be the translate of σ obtained by taking the origin at x. For $E \subset H$ let $\overline{d}^*(E,x)$, $\overline{d}^*_\theta(E,x)$ and $\overline{d}^*(E,x,\sigma)$ represent outer upper density of E at x, outer upper density of E at x in the direction θ, and outer upper density of E at x relative to the sector σ, respectively. If $f: H \to W$, where W is a topological space, then the cluster set $C(f,x)$ of f at x is the set of all $w \in W$ such that for every open subset U of W, containing w, x is a limiting point of the set $f^{-1}(U)$. The essential cluster set $C_e(f,x)$ of f at x is the set of all point $w \in W$ such that for every open set U of W, containing w, $\overline{d}^*(f^{-1}(U),x) > 0$. The directional (sectorial) cluster set $C(f,x,\theta)$ $(C(f,x,\sigma))$ and the directional (sectorial) essential cluster set $C_e(f,x,\theta)(C_e(f,x,\sigma))$ of f at x in the direction (sector) θ (σ) are defined analogously by restricting the relevant sets to $L_\theta(x)(\sigma(x))$.

Collingwood, Erdos and Piranian, Bagemihl, Bruckner and Goffman and others have established interesting properties of cluster sets and essential cluster sets. Extending Further these properties, we have obtained several other interesting properties of cluster sets and essential cluster sets. The results which are under our consideration are the following:

Collingwood in [4] and [5] proved

<u>Theorem</u> C. If F: H→W is continuous, where W is the Riemann sphere, and θ, $0<\theta<\pi$, is a fixed direction, then the set

$$\{x: x\in R; C(f,x) \neq C(f,x,\theta)\}$$

is of the first category on R.

<u>Theorem</u> C'. If f: H→W is continuous, where W is the Riemann sphere, then except a first category set of points on R, the set

$$\{\theta: 0<\theta<\pi; C(f,x) \neq C(f,x,\theta)\}$$

is of the first category in $(0,\pi)$.

Bruckner and Goffman [3] have proved these results for real valued functions. It is also known that the exceptional set of Theorem C may not be of measure zero (e.g. see [6 p.78]). Theorem C has further been advanced by Goffman and Sledd [9] by proving an analogue of this theorem in the case of essential cluster set which is as follows:

Theorem GS. If f: H→R is measurable and θ, $0<\theta<\pi$, is a fixed direction then except a set of points x of measure zero on R

$$C_e(f,x) \subset C_e(f,x,\theta)$$

If further, f is continuous then except a set of points x of the first category on R, this relation holds.

They have also shown that the equality between the sets $C_e(f,x)$ and $C_e(f,x,\ )$ may not hold at any point x on R. To supplement this result of Theorem GS, Belna, Evans and Humke in [2] have established Theorem BEH.

If f: H→W is measurable, where W is the Riemann sphere then almost every and nearly every x on R, the set

$$\{\theta:\ 0<\theta<\pi;\ C_e(f,x) \subset C_e(f,x,\theta)\}$$

has measure equal to π.

If, further, f is continuous then at almost every and nearly every x on R the above set is also of residual in (0,π) (Here 'nearly every x' means 'except a set of points x of the first category')

Mukhopadhyay in [13, Theorem 5] has proved a result which states as follows:

Theorem M. If f: H→R is measurable and θ_1, θ_2, $0<\theta_1<\pi, 0<\theta_2<\pi$, are two fixed directions then the set

$$\{x:\ x\in R;\ C_e(f,x,\theta_1) \not\subset C(f,x,\theta_2)\}$$

is of measure zero.

If, further, f is continuous then the above set is also of the first category.

Evans and Humke in [8] also have given a different proof of Theorem M and have also shown that the exceptional set of this theorem may not be countable.

Erdos and Piranan [7] have proved a theorem for arbitrary function which states as follows:

<u>Theorem</u> EP. Let F: $H \to W$ be arbitrary, where W is the Riemann sphere. Then except a first category set of points on R, the cluster set $C(f,x)$ of f at x is equal to the sectorial cluster set $C(f,x,\sigma)$ of f at x relative to any sector $\sigma \subset H$.

Bruckner and Goffman [3] enhanced this result by proving

<u>Theorem</u> BG. If f: $H \to R$ is arbitrary, then except a set of points x on R which is of the first category, the essential cluster set $C_e(f,x)$ of f at x is equal to the sectorial essential cluster set $C_e(f,x,\sigma)$ of f at x relative to any sector $\sigma \subset H$.

Goffman and Sledd [9] have shown that the exceptional set on R of the above theorem is also of measure zero. They have also shown in the same paper that the exceptional set of this theorem may be uncountable.

Bagemihl in [1] proved his remarkable theorem relating to the disjoint properties of arc cluster sets which is as follows:

<u>Theorem</u> B. If f: $H \to W$ is arbitrary, where W is the Riemann sphere, then the set of all points x on R at which there are two arcs γ_1 and γ_2 such that the sets $C_{\gamma_1}(f,x)$ and $C_{\gamma_2}(f,x)$ are disjoint is countable, ($C_\gamma(f,x)$ is the cluster set of f at x relative to the arc γ at x.)

In connection with Theorem M, we have proved in [10] the following theorems:

Theorem 1. If $f: H \to W$ is continuous, where W is any topological space having countable basis, and if ψ, $0<\psi<\pi$, is a fixed direction then the set

$$\{\theta: 0<\theta<\pi;\ C_e(f,x,\psi) \not\subset C(f,x,\theta)\}$$

is of the first category, except a set of points x on R which is of the first category and measure zero.

Theorem 2. If $f: H \to W$ is measurable, where W is any topological space having countable basis, and if ψ, $0<\psi<\pi$, is a fixed direction then the set

$$\{\theta: 0<\theta<\pi;\ C_e(f,x,\theta) \not\subset C(f,x,\psi)\}$$

is of measure zero at almost every $x \in R$.

Theorem 3. If $f: H \to W$, is continuous, where W is any topological space having countable basis, and if ψ, $0<\psi<\pi$, is a fixed direction then the set

$$\{\theta: 0<\theta<\pi;\ C_e(f,x,\theta) \not\subset C(f,x,\psi)\}$$

is void, except a set of points x on R which is a first category set of measure zero.

Corollary 1. If $f: H \to W$ is continuous, where W is any topological space having countable basis, then for each $x \in R$, except a first category set of measure zero on R, there exists a set of directions $\widetilde{\Theta}(x)$ which is residual in $(0,\pi)$ such that

$$\cap\ \{C(f,x,\theta);\ \theta \in \widetilde{\Theta}(x)\} \neq \emptyset.$$

Corollary 2. If $f: H \to W$ is measurable, where W is a second countable topological space and if $\psi \in (0, \pi)$ is a fixed direction, then for each $x \in R$, except a set of points x of measure zero on R, there exists a set of directions (x) which is of full measure in $(0, \pi)$ such that

$$\cup \{C_e(f,x,\theta): \theta \in (x)\} \subset C(f,x,\psi).$$

If further f is continuous, then for each $x \in R$, except a set of points x of the first category and measure zero on R

$$\cup \{C_e(f,x,\theta): 0 < \theta < \pi\} \subset C(f,x,\psi).$$

Relating to Theorem EP we have proved in [11]

Theorem 4. If $f: H \to W$ is arbitrary, where W is a second countable topological space, then except a first category set of measure zero on R,

$$C_e(f,x,\theta) \subset C(f,x,\sigma)$$

for every $\theta \in (0, \pi)$ and any sector $\sigma \subset H$.

Corollary 3. If $f: H \to W$ is measurable and $\{\sigma\}$ is a collection of sector in H, then the set

$$\{x: x \in R;\ C_s(f,x) \not\subset \cap\{C(f,x,\sigma): \sigma \in \{\sigma\}\}\}$$

is of measure zero.

If further, f is continuous then the above set is also of the first category. ($C_s(f,x)$ denotes the strong essential cluster set of f at x; for definition see [14]).

We have also studied some disjoint properties of cluster sets and essential cluster sets which are somewhat related to Bagemihl's theorem. The theorems which we have established in [11,12] in connection with Theorem B are as follows. (In all the results that follow, the topological space W is taken to be compact and normal having countable basis).

Theorem 5. If $f: H \to W$ is arbitrary, then the set of all points x on R at which there exist a direction $\theta_x, 0<\theta_x<\pi$, and a sector $\sigma(x)$ in H with the property that

$$C_e(f,x,\theta_x) \cap C(f,x,\sigma) = \emptyset$$

is countable.

Theorem 6. If $f: H \to W$ is arbitrary and if $\psi, 0<\psi<\pi$, is a direction then the set of points x on R at which there exists a direction θ_x $0<\theta_x<\pi$, such that

$$C_e(f,x,\theta_x) \cap C(f,x,\psi) = \emptyset$$

is of measure zero.

Theorem 7. If $f: H \to W$ is arbitrary then the set of all points x on R at which there exist a set $\Theta(x)$ of directions θ of positive outer measure and a direction $\theta_x \in (0,\pi)$ with the property that

$$C_e(f,x,\theta_x) \cap C(f,x,\theta) = \emptyset$$

for all $\theta \in \Theta(x)$ is countable.

If further, f is continuous then the set of all points x on R at which there exist a second category set of directions $\Theta(x)$ and a direction $\theta_x \in (0,\pi)$ such that

$$C_e(f,x,\theta_x) \cap C(f,x,\theta) = \emptyset$$

for all $\theta \in \Theta(x)$ is countable.

Theorem 8. If $f: H \to W$ is measurable then at every point x on R the set

$$\{\theta: 0<\theta<\pi;\ C_e(f,x) \cap C_e(f,x,\theta) = \emptyset\}$$

is of measure zero.

If further, f is continuous, then at every point x on R the above set is also of the first category.

This result is related to Theorem BEH.

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