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Lebesgue Equivalence

by

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We consider real functions on a closed interval [a,b]. Functions f and g are said to be <u>Lebesgue equivalent</u> if there is a homeomorphism h of [a,b] onto itself such that $g = f \circ h$, Lebesgue equivalence clearly satisfies the conditions of an equivalence relation. We shall be concerned with two sorts of questions:

(i) Does a given equivalence class contain a "nice" function?

(ii) Are all functions in a given equivalence class well behaved?

A related notion is that of Lebesgue equivalence of sets. Two sets S and T, contained in [a,b], are said to be <u>Lebesgue equivalent</u> if there is a homeomorphism h of [a,b] onto itself such that $T = h(S)$.

We shall discuss only matters of special interest to us. Some of these questions are related to functions whose Fourier series converge everywhere.

1. Our first remarks pertain to the well known theorem of Maximov, [13]. It is an elementary fact that every derivative is of class Baire 1 and has the

Darboux property, but that there are Baire 1 Darboux functions which are not derivatives. Maximov's theorem asserts that every Baire 1 Darboux function is Lebesgue equivalent to a derivative. The existing proof of this statement may be deemed unsatisfactory in one sense or another. The proof has been stated to be correct by some mathematicians. However, it is quite complicated and rather shadowy. In any case, a new and more transparent proof would be a welcome contribution. An independent treatment has been given by Choquet [5], for semicontinuous functions. Perhaps his proof works for the general case.

A related statement, also given by Maximov is that every Baire 1 Darboux function is Lebesgue equivalent to an approximately continuous function. The corresponding theorem on sets has been proved by Gorman, [11]. A point x is said to be a <u>bilateral c point</u> of a set S if, for every $c < x < d$, the sets $S \cap (c,x)$ and $S \cap (x,d)$ have cardinality c. Gorman's result asserts that if S is of type F_σ and each $x \in S$ is a bilateral c point of S then S is Lebesgue equivalent to a set T such that T has density 1 at each of its point. We note that if f is Baire 1 Darboux then, for each open set G, the set $f^{-1}(G)$ is of type F_σ and each of its points is a bilateral c point, while if f is approximately continuous then $f^{-1}(G)$ has density 1 at each of its points.

2. We now consider the following questions. Which functions are Lebesgue equivalent to everywhere differentiable functions? Which functions are Lebesgue equivalent to functions a) with summable derivative, b) with bounded derivative, c) with continuous derivative? If $f:[a,b] \to R$ is of bounded variation, then every function which is Lebesgue equivalent to f is also of bounded variation. Moreover, if f has a summable derivative then it must be of bounded variation. Accordingly, if f is Lebesgue equivalent to an everywhere differentiable function with summable derivative then f must be continuous and of bounded variation. We note, as a converse, that if f is continuous and of bounded variation then f is Lebesgue equivalent to an everywhere differentiable function with bounded derivative. The proof of the last statement appears in [4], and is an immediate consequence of a deep result of Zahorski [17]. To obtain the result we first observe that a continuous function of bounded variation is Lebesgue equivalent to a Lipschitzian function by using a slight modification of the arc length representation. The theorem of Zahorski which is applicable asserts that if $Z \subset [a,b]$ is of measure 0 and of type G_δ, and if $k > 0$, there is a homeomorphism h of $[a,b]$ onto itself, differentiable in the extended sense, with $h'(x) > k$ everywhere and $h'(x) = +\infty$ on Z. It follows that every

Lipschitzian function is Lebesgue equivalent to an everywhere differentiable function with bounded derivative.

The above remarks imply that the set of continuous functions of bounded variation is exactly the set of functions Lebesgue equivalent to either the everywhere differentiable functions with bounded derivative, or with summable derivative.

The Cantor function satisfies the above conditions and is accordingly Lebesgue equivalent to an everywhere differentiable function with bounded derivative. Since its derivative is zero on a dense set of intervals and cannot be everywhere zero the Cantor function is not Lebesgue equivalent to a continuously differentiable function. However, if $g:[0,1] \to R$ is the Cantor function and $f:[0,1] \to R$ is defined by $f(x) = g(x) + x$, then f is strictly increasing, and is accordingly Lebesgue equivalent to a linear function; hence, a continuously differentiable function.

Not every continuous function of bounded variation is Lebesgue equivalent to a continuously differentiable function, e.g., the Cantor function. The further necessary property involves a notion called point of varying monotonicity. A point x is a <u>point of varying monotonicity</u> of f if there is no neighborhood of x on which f is either strictly monotonic or strictly constant. For a Cantor like function its perfect set of support agrees with its set of points of varying monotonicity. Furthermore, the set of points of varying monotonicity is a closed set K and so has a unique decomposition $K = P \cup D$, where $P \cap D$ is empty, P is perfect and D is countable. P is the <u>perfect kernel</u> of K.

Our observation in [4] was that if f is to be Lebesgue equivalent to a continuously differentiable function then the set $f(K)$ must be of measure 0. This follows because for a continuously differentiable function the derivative is zero at every point in K, the set of points of varying monotonicity of f. If $f(K)$ had positive measure then $f'(x)$ would have to be positive at some points in K. Thus, a necessary condition that f be Lebesgue equivalent to a continuously differentiable function is that f be continuous, of bounded variation, and that the image $f(K)$ of its set K of points of varying monotonicity be of measure 0.

These conditions are also sufficient but the proof is rather delicate. We give a rough indication of the required construction. Let f satisfy the above conditions and let K, P, and D have the meanings given above. Then if $I_1, I_2, \ldots$ are the pairwise disjoint open intervals complementary to P, we have $\sum_n k_n < \infty$, where k_n is the variation $V(f, I_n)$ of f in the interval I_n. There is a homeomorphism h_1 of $[a,b]$ onto itself such that $\sum_n a_n < \infty$, where

$a_n = \frac{k_n}{|h_1(I_n)|}$, $|h_1(I_n)|$ being the length of the interval $|h_1(I_n)|$. We follow h_1 by a homeomorphism h_2 which acts only on the intervals $h_1(I_n)$, so that $f\circ(h_2\circ h_1)^{-1}$ is continuously differentiable on $h_2\circ h_1(I_n)$, has derivative equal to 0 at the end points, and $|f'(x)| \leq 4a_n$ for each $x \in h_2\circ h_1(I_n)$. This may be accomplished because $K \cap \bar{I}_n$ is closed and countable. It then follows from $f(K)$ has measure 0, that $f\circ(h_2\circ h_1)^{-1}$ is continuously differentiable. The interested reader will find the details in [4].

It is conjectured that the same conditions suffice for functions to be Lebesgue equivalent to functions in the class C^k, $k > 1$, or even in C^∞, but a required construction has not been attempted.

It is known that everywhere differentiable functions are of type VBG_*; details are in Saks [14]. As a converse, Fleissner and Foran have shown that every VBG_* function is Lebesgue equivalent to an everywhere differentiable function.

3. It is well known that there are continuous functions f on the unit circle, or on the interval $[-\pi,\pi]$, with $f(-\pi) = f(\pi)$, whose Fourier series do not converge everywhere. However, if a function is continuous and of bounded variation then its Fourier series converges uniformly. Since every function which is Lebesgue equivalent to a continuous function

of bounded variation is also continuous and of bounded variation, we have here a subset of the set $\mathcal{F}$ of functions whose Fourier series converge uniformly for all homeomorphic changes of variable. We shall discuss this set of functions as well as the set for which only everywhere convergence holds for all Lebesgue equivalent functions.

This treatment features the set of regulated functions. A function f is <u>regulated</u> if its right and left limits exist everywhere. A discussion of some properties of regulated functions is given in [8]. Also important in these considerations is the class of sets of absolute measure 0. A set S is of <u>absolute measure 0</u> if h(S) has Lebesgue measure 0 for every homeomorphism h. The regulated functions are invariant under Lebesgue equivalence as are the sets of absolute measure 0.

The main fact in this context is that if $f:[-\pi,\pi] \to R$ is such that the Fourier series of $f \circ h$ converges everywhere for every homeomorphism h of $[-\pi,\pi]$ onto itself, then f must differ from a regulated function on a set of absolute measure 0, or as we may say, f is AMZ-equivalent to a regulated function.

In the first place, if f is not AMZ-equivalent to a bounded function, then there is a homeomorphism h such that the Fourier series of $f \circ h$ does not converge everywhere. So we assume that f is AMZ-equivalent to a bounded function but not to any regulated function. By a standard normalization of f, we may assume that, for each $\delta > 0$, the sets $[f > 1] \cap (0,\delta)$ and $[f < -1] \cap (0,\delta)$ are not of absolute measure 0, A homeomorphism h of $[0,\pi]$ onto itself may be adjusted in such a way that $f\circ h > 1$ and $f\circ h < -1$ on a large enough part of enough pairs of intervals on which $\sin n_i t$ is positive in the one case and negative in the other, so that

$$\lim_{i \to \infty} \int_0^f f\circ g(t) D_{n_i}(t)\,dt = \infty$$ for an increasing sequence $\{n_i\}$. The homeomorphism h is defined on $[-\pi,0]$ to itself in such a way that the sequence

$$\left\{\int_{-\pi}^{0} f\circ g(t) D_{n_i}(t)\,dt\right\}$$ is bounded. This may be accomplished since f is bounded, by having $f\circ h$ nearly constant on enough pairs of intervals on which $\sin n_i t$ is positive on the one hand and negative on the other.

A stronger necessary condition is obtained by a similar but somewhat delicate analysis of the Dirichlet kernel. A necessary condition that $f:[-\pi,\pi] \to R$ be such that for every homeomorphism h, the Fourier series of $f\circ h$ converge everywhere is that it be AMZ-equivalent to a function g such that, for each $\varepsilon > 0$ and x there is a $\delta > 0$ such that, for each finite system of nonoverlapping intervals, $\{I_n\}$, indexed from right to left with $\cup I_n \subset (x,x + \delta)$, or indexed from left to right with $\cup I_n \subset (x - \delta, x)$, we have

$$\Sigma \frac{1}{n} |f(I_n)| < \varepsilon$$

where $f(I) = f(b) - f(a)$ for any interval $I = [a,b]$.

This condition is given in [9] where it is also shown to be sufficient. The proof is accomplished in analogy with an argument used by Salem, [15], in his proof of a criterion he gives for the convergence of a Fourier series.

There is an example of Lebesgue of a continuous function whose Fourier series converges everywhere but not uniformly, [18]. Baernstein and Waterman, [2], have shown that the present situation is similar. They have shown that for continuous functions f the Fourier series of $f \circ h$ converges uniformly for every homeomorphism h if and only if, for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each finite system $\{I_n\}$, indexed from left to right, or from right to left, with $\operatorname{diam}(\cup I_n) < \delta$ we have $\Sigma \frac{1}{n} |f(I_n)| < \varepsilon$. They then construct a continuous function f such that $f \circ h$ has everywhere convergent Fourier series, for each homeomorphism h, but for which the convergence is not always uniform.

4. We now consider some special classes of functions which are invariant under homeomorphisms. The set of functions of bounded variation has this

property since f and $f \circ h$ have the same variation. The other sets of functions we shall discuss are various sorts of extensions of the set of functions of bounded variation. First, for any regulated f we associate with each x the interval I_x whose end points are the right and left limits of f at x. For each y, let N(y) be the set of x for which $y \in I_x$. Banach's Theorem says that f is of bounded variation if and only if N(y) is summable.

We are now ready to consider three different generalizations of bounded variation.

a) Let $\Lambda = \{\lambda_n\}$ be a sequence of real numbers which are positive and strictly increasing to infinity. A regulated function f, with $f(x) = \frac{1}{2}\{f(x^+) + f(x^-)\}$, for every x, is said to be of Λ bounded variation, ΛBV, if

$$\sup \sum_{i=1}^{\infty} \frac{1}{\lambda_i} |f(x_i) - f(y_i)| < \infty$$

for pairwise disjoint sequences $\{(x_i, y_i)\}$ of intervals. Of particular interest is the case $\Lambda = \left\{\frac{1}{n}\right\}$. A function of bounded Λ variation for this case is said to be of harmonic bounded variation, HBV. These notions were introduced in [16], the notion HBV being implicit in [7]. If $f \in$ HBV, then the Fourier series of f converges everywhere, [7]. The proof follows immediately from the change of variables theorem of the last section. It may be proved directly by an argument similar to the one used in the proof of that theorem, but somewhat simpler. A third proof, [16], may be given by use of the standard Lebesgue test for convergence. Conversely, Waterman showed, [16], if Λ is such that $\Lambda BV \supset HBV$ properly, then there is a function in ΛBV whose Fourier series diverges somewhere.

These results suggest the guess that HBV is a characterization of the set of function whose Fourier series converge everywhere for all changes of variables. However, this conjecture seems to be difficult to handle. We feel that this is a worthwhile problem to consider.

b) Let $\Phi:[0,\infty) \to [0,\infty)$, with $\Phi(0) = 0$, $\lim_{x \to 0} \Phi(x) = 0$, and $\lim_{x \to \infty} \Phi(x) = \infty$, be a convex function, and let Ψ be the associated function in the sense of W. H. Young. Let $[a,b]$ be an interval. For each partition $\pi: a = x_0 < x_1 < \dots < x_n = b$, let $V_\Phi(f,\pi) = \sum_{i=1}^{n} \Phi(|f(x_i) - f(x_{i-1})|)$. The Φ variation of f is defined by $V_\Phi(f) = \sup V_\Phi(f,\pi)$, for all partitions π of $[a,b]$. Now, if f is of bounded Φ variation and $\Sigma\Psi\left(\frac{1}{n}\right) < \infty$, then the Fourier series of f converges everywhere. Moreover, [1], if $\Sigma\Psi\frac{1}{n} = \infty$ there is an f of bounded Φ variation whose Fourier series diverges somewhere. The positive part of this result follows from the fact (an immediate consequence of the Young inequality) that HBV contains the set of functions of bounded Φ variation if $\Sigma\Psi\left(\frac{1}{n}\right) < \infty$.

c) This extension of bounded variation follows from the Banach criterion. Let w be a continuous increasing function on $[0,\infty)$ such that $w(0) = 0$ and $\lim_{x \to \infty} w(x) = \infty$. Let $S(w)$ be the set of functions for which $\int w(N(y))dy < \infty$. Garsia and Sawyer, [6], showed that for $w(x) = \log^+(x)$, $f \in S(w)$ implies that the Fourier series of f converges everywhere, i.e., if f is such that $\int \log^+ N(y)dy < \infty$, then the Fourier series of f converges everywhere. This result also follows readily, as shown in [7], from the result in a). In this case, a simple lemma on measurable sets is needed. Let $I = [a,b]$, and let $A_n \subset I$, $n = 1,2,\ldots$, with $I = \cup A_n$, be such that each $x \in I$ is in a finite number of the A_n. For each n, let S_n consist of those x which are in exactly n of the A_i. Then $\sum n \cdot m(A_n) = \infty$ implies $\sum \log n \cdot m(S_n) = \infty$. Conversely, if $w(x)$ is such that $\lim_{x \to \infty} \frac{\log^+(x)}{w(x)} = \infty$ there is an $f \in S(w)$ whose Fourier series diverges somewhere.

Thus, each of the above three variations criteria is sharp.

Brownian motion is of bounded Φ variation, with probability 1, for $\Phi(x) = x^{2+\varepsilon}$, $\varepsilon > 0$. It follows that the probability is 1 that the Fourier series of a continuous function converges uniformly.

5. We now turn to questions of the sort: Does a Lebesgue equivalence class contain a nice function? An example is the Bohr-Pal theorem which says that every periodic continuous function, of period 2π, is Lebesgue equivalent to a function whose Fourier series converges uniformly. The only known proof of this theorem involves some rather sophisticated complex variable theory, including the Riemann mapping theorem, the Caratheodory theorem extending the mapping to the boundary in case it is a Jordan curve, and the fact that convergence of the power series expansion of the mapping function extends to the boundary. The last fact uses Fejer's theorem on (C,1) convergence of Fourier series together with the fact that the area of the image domain is given by a certain series. The convergence of this last series turns out to be a Tauberian condition for (C,1) convergence. An outline of the proof is given in [10]. The special character of this proof has made the study of analogous questions impossible.

There are two indicated directions in which extensions may possibly be made. First, it is natural to ask whether similar results hold for other orthonormal systems. Such results would require a new kind of proof not using complex variables. Little progress seems to have been made in this direction. The other problem is to determine which measurable functions f have a Lebesgue equivalent $f \circ g$ whose Fourier series converges everywhere. Of course, in order to have this property, $f \circ g$ must be equal almost everywhere to a Baire 1 function. We are accordingly interested in knowing whether or not every measurable function is Lebesgue equivalent to a function which is almost everywhere equal to a Baire 1 function. Interesting results along these lines were obtained by Gorman, [12], and subsequent results by Bruckner, Davies, and Goffman, [3].

The positive results obtained by Gorman are: a) If f has the property of Baire there is a homeomorphism g such that $f \circ g$ is almost everywhere equal to a Baire 1 function. b) If f is measurable and its range is finite, there is a homeomorphism g such that $f \circ g$ is almost everywhere equal to a Baire 1 function.

On the other hand, Gorman gave an example of a measurable f such that for no homeomorphism g is $f \circ g$ equal almost everywhere to a Baire 1 function. Since a function whose Fourier series converges everywhere must be equal almost everywhere to a Baire 1 function, this shows the existence of a measurable function which is not Lebesgue equivalent to any function whose Fourier series converges everywhere.

The example of Gorman turns out not to be absolutely measurable, i.e., it has a non measurable

Lebesgue equivalent function. It is shown in [3] that if f is absolutely measurable there is a Lebesgue equivalent $f \circ g$ which is almost everywhere equal to a Baire 1 function.

The proof uses the following characterization: f is Lebesgue equivalent to a function $f \circ h$ which is almost everywhere equal to a Baire 1 function if and only if there is a c-dense F_σ type set E and a Baire 1 function g such that $f(x) = g(x)$ for every $x \in E$. We accordingly must show that absolutely measurable functions have this characterizing property. This is shown by an iteration procedure using first the proof of the theorem for the case where f takes on countably many values, a lemma that says that if f is absolutely measurable and P is perfect, then there is a perfect $Q \subset P$ such that f restricted to Q is continuous, and the fact that the limit of a uniformly convergent sequence of Baire 1 functions is also Baire 1.

Because of this result, we do not know of any absolutely measurable function which is Lebesgue equivalent to a function whose Fourier series diverges at some point. We do not even know if every regulated f is Lebesgue equivalent to a function whose Fourier series converges everywhere.

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