QUERIES

Queries will continue to be listed until solutions are received by the EXCHANGE. Solutions received within the past six months appear immediately following the list of queries.

1. If $f:R \to R$ is continuous with respect to the density topology on both the domain and range, must the image of a set of measure zero be of measure zero?

2. If $f:R \to R$ is continuous must the upper symmetric derivate of f be equal to the upper approximate symmetric derivate of f on a residual set?

3. If $f:R \to R$ is continuous on the set E and $|x-f(x)| = 1$ for each x in E, can $\dim(E) < \dim(f(E))$, where dim() denotes Hausdorff dimension?

4* Can Query 4, parts b and c, be answered without the use of the continuum hypothesis?

7. If $f:R \to R$ is continuous and its graph is sparse, must f'_{ap} exist a.e.?

8. Let Λ denote the Hausdorff one-dimensional measure. If $E \subset R_2$ and $\Lambda(E)>0$, must $\{\vec{p}:\vec{p}=\vec{r}+\vec{s},\ \vec{r}\varepsilon E,\ \vec{s}\varepsilon E\}$ contain an arc?

10. If $S \subset R$ is nowhere dense and $\dim(S)=\alpha$ $(0 \leq \alpha \leq 1)$, is there a perfect set T such that $\dim(T) \geq 1-\alpha$ and $S \oplus T$ is nowhere dense? Here, dim() denotes Hausdorff dimension.

11. *Proposed by Jan Mycielski, Department of Mathematics, University of Colorado, Boulder, Colorado 80302.*
Let ρ be a distance function over the real line R which generates the usual topology and is translation invariant, i.e., $\rho(x+t, y+t) = \rho(x,y)$ for all x,y,t in R. Let $A,B \subset R$ be two Borel sets which are isometric relative to ρ, i.e., there exists a bijection $f:A \leftrightarrow B$ such that $\rho(f(x),f(y)) = \rho(x,y)$ for all x,y in A. Must the Lebesgue measures of A and B be equal? (If A and B are open sets, then the answer is yes, see Jan Mycielski, Remarks on invariant measures in metric spaces, Coll. Math. 32 (1974), 105-112.)

Query 4.

Let x and y be two points in R_2 and let [xy] denote the closed line segment joining x to y. For $E \subset R$, let $[E] = \{p\varepsilon R_2 : p\varepsilon[xy],\ x\varepsilon E \times \{0\},\ y\varepsilon E \times \{1\}$.

a) Does $|E| = 0$ imply that the planar measure of [E] is 0?

b) Does the measurability of E imply the measurability of [E]?

c) If E is measurable and $\varepsilon>0$, is there a compact set $F \subset E$ such that $|[E]| < |[F]| + \varepsilon$?

Solution by

Real Analysis Exchange Editorial Staff.

a) Let P denote the Euclidean plane with a rectangular Cartesian coordinate system where the x-axis is horizontal and the vertical axis is the y-axis. Let R be the closed parallelogram (with Interior) in P of height 1 and width w whose lower left vertex is at (0,0). Suppose the top T (bottom B) of R is divided into three abutting closed segments T_1, T_2, and T_3 (B_1, B_2, and B_3) of equal length where the second is central in each case. Let R_{ij} denote that closed parallelogram in R whose bottom segment is B_i and whose top segment is T_j. Then it is easy to see that

(1) $\{(x,y)\varepsilon R : 1/4 \leq y \leq 3/4\} \subset \bigcup R_{ij} \quad (i,j\varepsilon\{1,3\})$.

Using the notation introduced above, we define the set mapping

(2) $$\Delta(R) = \{R_{ij}: i,j\varepsilon\{1,3\}\}.$$

Now, if R^1 denotes the closed unit square, we define

(3) $$R^2 = \Delta(R^1), \text{ and inductively,}$$

$$R^{n+1} = \cup\Delta(R) \qquad (R \varepsilon R^n).$$

Finally, we define R^{n*} to be the union of the parallelograms composing R^n. The collection $\{R^{n*}: n=1,2,\ldots\}$ is a decreasing sequence of compact sets and it is easily seen that

(4) $$\bigcap_{n=1}^{\infty} R^{n*} = [C]$$

where C denotes the Cantor ternary set. But, from (1) it follows that

(5) $$S = \{(x,y)\varepsilon R^1 : 1/4 \leqq y \leqq 3/4\} \subseteq [C]$$

and as such $|[C]| \geqq 1/2$.

b) Suppose that $2^{\aleph_0} = \aleph_1$, let R^1 denote the closed unit square, and let C denote the Cantor ternary set. Let

$$\mathcal{F} = \{F_0, F_1, \ldots, F_\alpha, \ldots : \alpha<\omega_1\}$$

be an enumeration of the perfect sets of positive measure in $R^1 \cap \{(x,y): 1/4 \leqq y \leqq 3/4\} = S$, where ω_1 is the initial ordinal of power $\aleph_1$. Similarly, let

$$\mathcal{K} = \{K_0, K_1, \ldots, K_\alpha \ldots : \alpha<\omega_1\}$$

be an enumeration of the perfect subsets of C. We will simultaneously define two complementary subsets S_1 and S_2 of S and also two complementary subsets C_1 and C_2 of C as follows.

0. Let $s_0^1 \varepsilon F_0$. As $s_0^1 \varepsilon S$, there are two points c_0^1 and c_0^2 in C such that $s_0^1 \varepsilon [(c_0^1,0),(c_0^2,1)]$. As F_0 is a perfect set of positive measure, there is an $s_0^2 \varepsilon F_0$ such that

$$s_0^2 \notin \bigcup[(c_0^i,0),(c_0^j,1)] \quad (i,j\varepsilon\{1,2\}).$$

Also, as K_0 is a perfect set, there is a point k_0 such that that $k_0 \neq c_0^i$ $(i=1,2)$ and yet $k_0 \varepsilon K_0$.

1. Suppose that s_α^1, s_α^2, c_α^1, c_α^2, and k_α have been defined for $\alpha<\beta$ such that

a. s_α^1 and $s_\alpha^2 \varepsilon F_\alpha$,

b. $s_\alpha^1 \varepsilon [(c_\alpha^1,0),(c_\alpha^2,1)]$,

c. $s_\alpha^2 \notin \bigcup [(c_\gamma^1,0),(c_\sigma^2,1)]$ $(i,j\varepsilon\{1,2\}$ and $0 \leqq \sigma \leqq \gamma \leqq \alpha)$,

d. $k_\alpha \varepsilon K_\alpha$,

e. $c_\gamma^i \neq k_\sigma$ for $\gamma,\sigma \leqq \alpha$ and $i\varepsilon\{1,2\}$.

Now, as F_β is a perfect set of positive measure, there is a point s_β^1 in $F_\beta \cap S - [E]$ such that $s_\beta^1 \neq s_\alpha^2$ $(0 \leqq \alpha < \beta)$. As $s_\beta^1 \notin [E]$ there are points c_β^1 and c_β^2 in C such that $s_\beta^1 \varepsilon [(c_\beta^1,0),(c_\beta^2,1)]$ but $c_\beta^i \neq k_\alpha$ $(0 \leqq \alpha < \beta$ and $i\varepsilon\{1,2\})$. Also, as β is an enumerable ordinal,

$$\bigcup[(c_\gamma^i,0),(c_\sigma^j,1)] \qquad (0 \leqq \gamma \leqq \sigma \leqq \beta \text{ and } i,j\varepsilon\{1,2\})$$

is of measure zero and consequently, there is another point s_β^2 εF_βsuch that $s_\beta^2 \not\in$ $[(c_\gamma^i,0),(c_\sigma^j,1)]$ $(0 \leqq \gamma \leqq$ $\leqq \sigma \leqq \beta$ and $i,j\varepsilon\{1,2\})$. Finally, let k_β ε $K_\beta - \{c_\alpha^i : 0 \leqq \alpha \leqq$ $\leqq \beta$ and $i\varepsilon\{1,2\}$. This completes the induction and we define

1. $S_1 = \{s_\alpha^1 : 0 \leqq \alpha < \omega_1\}$,
2. $S_2 = \{s_\alpha^2 : 0 \leqq \alpha < \omega_1\}$,
3. $C_1 = \{c_\alpha^i : 0 \leqq \alpha < \omega_1$ and $i=1,2\}$,
4. $C_2 = \{k_\alpha : 0 \leqq \alpha < \omega_1\}$.

Now, as $C_1 \subsetneq C$ it follows that C_1 is of measure zero. However, $[C_1]$ intersects every perfect set of positive measure in S and also intersects the complement of every perfect set of positive measure in S. It follows that $[C_1]$ S and hence $[C_1]$ is nonmeasurable. This answers Query 4b.

c) To answer 4c, let K be a closed set in C_1. If K were uncountable, K would contain a perfect set in C. But $C_2 \cap C_1 = \emptyset$ and C_2 intersects every perfect set in C. Consequently, K is countable and [K] has measure zero for every closed set K in C_1. It follows that 4c is to be answered in the negative even if the double bar is interpreted to mean outer measure in the statement of 4c.

NOTE. 'Regarding problem 4a. Professor Davies (Leicester) points out that the solution is, in essence, obtained in his paper "On accessibility of plane sets and differentiation of functions of two real variables", Proceedings, Cambridge Philosophical Society, 1952. In this paper one sees that a set L of lines can be constructed in P that covers the open first quadrant but whose union meets the lines y=0 and y=1 in sets of linear measure zero. If these sets are $E_1 \times \{0\}$ and $E_2 \times \{1\}$ then the set $E = E_1 \cup E_2$ provides a best (worst) possible solution to the problem.

Query 5.

Characterize the countable G_δ subsets of R.

Solution by

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(Galvin's solution) The countable G_δ subsets of R can be characterized as the scattered sets, and the same is true for countable G_δ subsets of any complete separable metric space. Recall that a set is scattered if every nonempty subset has an isolated point. We first prove

1. In a metric space, every scattered set is a G_δ.

The characterization will then follow from the following well known results concerning scattered sets.

2. In a Baire space, every countable G_δ set is scattered.
3. In a hereditarily Lindelöf space, every scattered set is countable.

Proof of 1.

Let $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\xi \subseteq \cdots$ $(\xi<\omega_1)$ be the Cantor-Bendixson derivatives of a set A. For each $a \in A$ there is an ordinal $\xi(a)$ such that $a \in A_{\xi(a)} - A_{\xi(a)+1}$, i.e., a is an isolated point of $A_{\xi(a)}$. Choose $\varepsilon(a)$ such that $0<\varepsilon(a)<1$ and

$$S(a,\varepsilon(a)) \cap A_{\xi(a)} = \{a\}, \text{ where } S(a,r) = \{x:d(x,a)<r\}.$$

Then,

$$A \quad \bigcap_{n=1}^{\infty} U_n \quad \text{where} \quad U_n = \bigcup_{a\in A} S(a,\varepsilon(a)/n).$$

Now, let $x \in \bigcap_{n=1} U_n$, and choose $a \in A$ such that $x \in S(a, \varepsilon(a)/2)$ and $\xi(a)$ is as small as possible. Let $\varepsilon>0$ and choose $n>2$ such that both $1/n<\varepsilon$ and $1/n<\varepsilon(a)/2$. As $x\in U_n$, there is a $b\in A$ such that x $S(b,\varepsilon(b)/n)$. But then,

$$d(a,b) < d(a,x)+d(x,b) < \varepsilon(a)/2 + \varepsilon(b)/n < \varepsilon(a).$$

It follows from the definition of $\varepsilon(a)$ that $a = b$ and hence that the distance between x and a is less than ε. As ε was arbitrary we can conclude that x=a and the proof is complete.

Proof. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences in I_ε such that $x_n \uparrow x_o$ and $y_n \downarrow x_o$. If $x_o < z < x_o + \varepsilon$, then choose n such that $x_o < y_n < z$ and so obtain that $f(x_o) < f(y_n) < f(z)$. Similarly, if $x_o - \varepsilon < z < x_o$, then $f(z) < f(x_o)$. All but countably and below. Hence, $\overline{I}_\varepsilon \setminus I_\varepsilon$ is countable and so I_ε is a G_δ.

Note. If one sets

$$J_\varepsilon = \{x \varepsilon R: \text{ for } 0 < x^* - x < \varepsilon,\ \operatorname{sgn}(x^* - x) = \operatorname{sgn}(f(x^*) - f(x))\},$$

then J_ε need not be measurable. Indeed, let E be an arbitrary subset of R and define

$$f(x) = \begin{cases} \operatorname{Tan}^{-1}(x) & \text{if } x \varepsilon E \\ \pi/2 & \text{if } x \notin E. \end{cases}$$

Then $J_\varepsilon = E$ for every $\varepsilon > 0$.