Real Analysis Exchange Vol. 2 (1976)

Richard J. O'Malley, Department of Mathematics,
University of Wisconsin-Milwaukee
Milwaukee, Wisconsin 53201

Approximately Differentiable Functions: The r Topology

Rather than treating this article as an elongated abstract the author feels it would be more in the spirit of the Real Analysis Exchange to trace the development of the paper. The discussion is deliberately informal.

The problem first originated in the study of approximate derivatives. From joint work with Clifford Weil, [5], it became apparent that if an approximate derivative was equal to zero in every neighborhood where it was a derivative then it must be identically zero. Stated differently this implies that if f is an approximately differentiable function which is constant in every neighborhood where it is differentiable then f must be identically a constant. Thus it is clear that if an approximately differentiable function is zero in a dense set it is always zero.

Now switching to the density topology [2] there are two interesting facts. First in this topology every set of measure zero is closed. Second given any density closed set X and Euclidean closed set Y, with $X \cap Y = \phi$, there is an approximately continuous function g with $0 \leq g \leq 1$, $g(X) = 0$ and $g(Y) = 1$. Thus given any countable dense set X, such as the rationals, and a point

x_0, not in X, there is an approximately continuous function separating them. However the discussion above shows there is not an approximately differentiable function doing the same job.

Now the density topology was generated to have exactly as many open sets as were necessary to make approximately continuous functions continuous. The above seemed to the author to point out that the approximately differentiable functions did not require as many sets be added to the Euclidean topology. Thus the question became "Is the density topology the coarsest topology for which the approximately differentiable functions are continuous?".

Assuming the answer was no the author had to decide what open sets to drop out. Alternately this leads to the consideration of how the structure of approximately differentiable functions differs from that of approximately continuous functions. Further the question came up to the effect that "If the density topology is the wrong one can it be expected that, as with the density topology, the correct topology, r, will have the property that f is r-continuous if and only if f is approximately differentiable?"

This last question is answerable even before the others. Obviously every ordinary continuous function would still be continuous in the new topology and, since these functions are not all approximately differentiable, the answer is no. Therefore if there is a different topology to be found it would need a new subclass of approximately continuous functions properly containing the approximately differentiable.

One property that had been useful in studying approximately differentiable functions suggested itself naturally as perhaps the device needed to form this new subclass. Namely in [6] Tolstoff shows that if f is approximately differentiable then every perfect set contains a portion on which f is continuous. The author took this property and called functions having it Baire *1. Then the question becomes, "Is the density topology the coarsest topology making the approximately continuous Baire *1 functions continuous?".

In turn this led the author to study Baire *1 Darboux functions f [14]. They were found to have the interesting property, NEI, that for every a $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ have nonempty Euclidean interior. Therefore returning to approximately continuous Baire *1 functions the author looked at $A = \{S: \exists$ f and a such that $S = [f > a]$ and f is approximately continuous Baire *1$\}$. The collection A forms a subbasis for the desired topology; it was not hard to show that A actually formed a basis. Therefore since the NEI property would be preserved under arbitrary unions the main question was answered. The density topology was not the proper topology, which the author called r. However this in no way actually told what the r topology was. That is saying that r is the topology generated by using A as a basis doesn't say much about r. In trying to give alternate characterizations to what sets were in A, without reference to a function f, the author observed that all the sets had to be F_σ's because the functions were of Baire class 1. However it could also be shown that they had to be G_δ. Thus it was logical to study the class of sets which were both F_σ's and G_δ's at the same time. Historically it seems that these sets are called ambiguous of class 1, or resolvable [3], or bivalent [1]. The author took the liberty of calling such sets ambivalent (partly as a compromise, partly due also to the feeling that study of these sets generates after awhile.) Functions having $\{x: f(x) > a\}$ and $\{x: f(x) < a\}$ ambivalent for every a were naturally called ambivalent.

Finally it was shown that the r topology is the one generated by the basis of sets $B = \{u: u \text{ is density open and ambivalent}\}$. Therefore it seems that the sets which are open in the density topology but not ambivalent have to be dropped out. However this is not exactly true. There are r-open sets which are not ambivalent. It is still an open question exactly which are the r-open sets. Further, though this is not explicitly done in the paper, the r-continuous functions are not precisely the approximately continuous ambivalent functions. Thus exactly which are the r-continuous functions is also an open question. On the positive side it is shown that the r-topology is "near" normal in the sense that if X is a r-closed set and Y is a Euclidean closed set with $X \cap Y = \phi$, then there is an approximately differentiable function g with $0 \leq g \leq 1$ and $g(X) = 0$ and $g(Y) = 1$.

References

[1] J. Dugundji, Topology, Allyn and Bacon, page 92.

[2] C. Goffman, C. J. Neugebauer and T. Nishiura, Density topology and approximate continuity, Duke Math. J. (1961) pp 497-505.

[3] Kuratowski, Topology I and II.

[4] R. J. O'Malley, Baire *1, Darboux Functions, to appear in PAMS.

[5] R. J. O'Malley and C. Weil, The oscillatory behavior of certain derivatives, submitted to TAMS.

[6] G. Tolstoff, Sur la derive approximative exacte, Mat. Sb. 4(1938) pp. 499-504.

Received October 15, 1976