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The Darboux and Denjoy Properties for Directional Derivatives and Gradients

The purpose of this article is to inform readers of The Exchange of some results concerning directional derivatives and gradients that have never been published. Some are due to the author while others appear in the Ph.D. thesis of Raymond Peter Goedert which was written under the author's direction. They deal with extensions of the Darboux and Denjoy properties of derivatives of real valued functions of a real variable to functions of several variables. The definition of the latter property for such functions is immediate. In fact, if the domain of f is a measure space and the range, a topological space, then f has the Denjoy property if for each open set U, $f^{-1}(U)$ either is empty or has positive measure. The Darboux property has no one natural extension to functions of several variable. Consequently, early research in the area centered on finding a Darboux-type property that would be possessed by a predetermined class of functions. For example, Neugebauer (see [2]) define Darboux sets and then proved that a partial derivative of a linearly continuous function maps Darboux sets to connected sets. The Bruckners (see [1]) noticed the common feature of these concepts and defined the notion of Darboux $(\mathcal{B})$ where $\mathcal{B}$ is a basis of connected sets for the topology of the domain by insisting that the image of each element of $\mathcal{B}$ be a connected set. Thus, the problem became one of determining conditions on a basis $\mathcal{B}$ so that a particular class of functions is Darboux $(\mathcal{B})$.

One such problem was motivated by Neugebauer (see [3]) who proved that $f: E_1 \to E_1$ is a Darboux function of Baire class one if and only if for each a, $\{x: f(x) \geq a\}$ and $\{x: f(x) \leq a\}$ are G_δ sets which contain the endpoints of any interval they contain. This last condition on the sets must be defined in an arbitrary topological space relative to a basis $\mathcal{B}$.

Definition: A set S is $\mathcal{B}$ closed if for each $U \in \mathcal{B}$, $U \subset S$ implies $C\ell\, U \subset S$.

Next the conditions on $\mathcal{B}$ are given under which a version of Neugebauer's Theorem can be proved.

Definition: A basis $\mathcal{B}$ of a metric space satisfies condition (1) if for each open ball B and each $x \in \mathrm{Bd}\, B$, there is a $U \in \mathcal{B}$ such that $U \subset B$ and $x \in \mathrm{Bd}\, U$. Also $\mathcal{B}$ satisfies condition (2) if for each $U \in \mathcal{B}$ and $x \in C\ell\, U$, there is a $V \in \mathcal{B}$ such that $C\ell\, V - \{x\} \subset U$.

Theorem (Goedert): Let $f: E_n \to Y$ where Y is a separable metric space, be a function of Baire class one and let $\mathcal{B}$ be a basis of E_n satisfying (1) and (2) with each $U \in \mathcal{B}$

connected. Then f is Darboux ($\mathcal{B}$) if and only if for each closed set $K \subset Y$, $f^{-1}(K)$ is $\mathcal{B}$ closed.

In [1] there is established another characterization of functions of Baire class one which are Darboux ($\mathcal{B}$) which was inspired by a theorem due to Zahorski. He proved that a function $f: E_1 \to E_1$ is of Baire class one having the Darboux property if and only if for each open set U, $f^{-1}(U)$ is a dense-in-itself F_σ set. As before dense-in-itself must be defined.

<u>Definition:</u> Let $\mathcal{B}$ be a basis for a topological space. A set S is dense-in-itself ($\mathcal{B}$) if for each $x \in S$ and $U \in \mathcal{B}$ with $x \in C\ell\, U$, $S \cap U$ contains a point other than x.

<u>Theorem</u> (Goedert): Let Y be a separable metric space and $f: E_n \to Y$ be of Baire class one. Let $\mathcal{B}$ be a basis of E_n satisfying (1) and (2). Then f is Darboux ($\mathcal{B}$) if and only if for each open set U, $f^{-1}(U)$ is dense-in-itself ($\mathcal{B}$).

In [1] this theorem is proved under conditions on $\mathcal{B}$ which are similar to (1) and (2) but neither imply nor are implied by them.

The main problem Goedert dealt with in his thesis was to find a condition on $\mathcal{B}$ so that the gradient of every differentiable function is Darboux ($\mathcal{B}$). It is first observed that no such condition is possible if it is not assumed that f is differentiable. For example let

$$f(x,y) = \begin{cases} 2xy(x^2+y^2)^{-1/2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It is easily shown that $f \in C^\infty(E_2 - \{(0,0)\})$ and hence differentiable on $E_2 - \{(0,0)\}$. But f is not differentiable at $(0,0)$ even though $\text{grad } f(0,0) = (0,0)$. In addition $|\text{grad } f(x,y)| \geq 1$ for $(x,y) \neq (0,0)$; so grad f maps any neighborhood of $(0,0)$ onto a subset of E_2 having $(0,0)$ as an isolated point and consequently cannot possibly be connected.

Definition: Let $S \subset E_n$. A point $x \in C\ell\, S$ is called accessible from S if there is an $\alpha > 0$ and a sequence $\{c_n\}$ with $c_n \in S - \{x\}$ for each n and $\lim_{n\to\infty} c_n = x$ such that for each n the open ball with center c_n and radius $\alpha|x - c_n|$ is contained in S. The set S has an accessible boundary if each x in its boundary is accessible from S.

Roughly speaking, a set S has an accessible boundary if it does not come to a cusp on its boundary.

Theorem (Goedert): Let $\mathcal{B}$ be a basis of connected sets for E_n, each element of which has an accessible boundary. If $f: E_n \to E_1$ is differentiable at x for each $x \in E_n$, then grad f is Darboux $(\mathcal{B})$.

Two assertions follow easily from the above theorem. Each was obtained for the ball basis of E_n by the author.

<u>Corollary (Goedert)</u>: Let $\mathcal{B}$ and f be as in the previous theorem and let $\nu \in E_n$, $|\nu| = 1$. The directional derivative of f in the direction ν is Darboux ($\mathcal{B}$). Also $|\text{grad } f|$ is Darboux ($\mathcal{B}$).

In his thesis Goedert did not show that the accessible boundary condition was necessary but he did give an example of a set, S, with $(0,0)$ as a nonaccessible boundary point and a differentiable function f with $\text{grad } f(0,0) = (0,0)$ while $\partial_x f(x,y) = 1$ for each $(x,y) \in S$. Thus $\text{grad } f(C\ell\, S)$ is not connected.

With no restrictions on the function, the existence of the directional derivative, $\partial_\nu f$, of f in the direction ν need not imply that $\partial_\nu f$ has the Denjoy property. For example let

$$f(x,y) = \begin{cases} x & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \,. \end{cases}$$

Then $\partial_x f(x,0) = 1$ while $\partial_x f(x,y) = 0$ if $y \neq 0$. So $(\partial_x f)^{-1}(0,2)$ is the x-axis which has measure 0. The next two theorems state conditions on f sufficient to imply that directional derivatives have the Denjoy property.

Theorem: Let $f\colon E_n \to E_1$ and let $\nu \in E_n$, $|\nu| = 1$, and suppose that $\partial_\nu f(x)$ exists for each $x \in E_n$. For each $x \in E_n$ write $x = (x_1, x_1')$ where x_1 is the component of x in the direction ν, $x_1' \in E_{n-1}$ and $x = x_1\nu + x_1'$. If $\{x_1\colon f(x_1, x_1')$ is continuous in $x_1'\}$ is dense in E_1, then $\partial_\nu f$ has the Denjoy property.

Theorem: Let f and ν be as above. If f is linearly continuous, then $\partial_\nu f$ has the Denjoy property.

Concerning the gradient an example has already been given of a continuous function which is differentiable everywhere except at $(0,0)$ where both partial derivatives exist and are 0, but whose gradient has length at least 1 except at $(0,0)$. This gradient does not have the Denjoy property. The major unsolved problem in this area of research is whether or not the gradient of a differentiable function has the Denjoy property.

References

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