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The Distance Set of Certain Cantor Sets

Let A be a subset of Euclidean n-space. The distance set of A is the set $D(A) = \{|P-Q|: P,Q \in A\}$, where $|P-Q|$ denotes the Euclidean distance between P and Q. In 1917, Hugo Steinhaus [8] proved that the distance set of the Cantor ternary set $C_{1/3}$ is the interval $[0,1]$. Three years later, he [9] demonstrated that the distance set of any set with positive Lebesgue measure contains a non-degenerate interval with left endpoint zero. In light of these results, we make the following definitions: the set A is called a Steinhaus set (S-set) if $D(A)$ contains a non-degenerate interval with left endpoint zero; furthermore, if this interval has length equal to the diameter of A, then A is called a complete Steinhaus set (CS-set).

For $0 < \lambda < 1$, the Cantor "middle λ-th" set is constructed as follows: From the closed unit interval delete the open middle λ-th segment $((1-\lambda)/2,(1+\lambda)/2)$ leaving the two closed intervals $A_{11} = [0,(1-\lambda)/2]$ and $A_{12} = [(1+\lambda)/2,1]$ each of length $s_1 = (1-\lambda)/2$. From A_{11} and A_{12} delete the open middle segments of length λs_1 leaving 2^2 closed intervals each of length $s_2 = [(1-\lambda)/2]^2$. Denote these closed intervals, from left to right, by A_{21}, A_{22}, A_{23}, and A_{24}. Continue this process inductively. The n-th stage of the construction consists of deleting the open middle λ-th segment of length λs_{n-1} from each of the 2^{n-1} closed intervals that result from the (n-1)-th stage to obtain 2^n closed intervals $A_{n1}, A_{n2},\ldots,A_{n2^n}$ each of length $s_n = [(1-\lambda)/2]^n$.

For each index n, set $A_n = \bigcup_{i=1}^{2^n} A_{ni}$. Then

$$C_\lambda = \bigcap_{n=1}^{\infty} A_n$$

In this note, we give an account of the results which led to the determination of the set of values λ for which the sets C_λ and $T_\lambda \equiv C_\lambda \times C_\lambda$ are CS-sets.

§2. The set C_λ.

In 1955, N. C. Bose Majumder and H. M. Sen Gupta [1] proved that C_λ is a CS-set for $\lambda = 1/(2n+1)$ $[n=1,2,\ldots]$; in 1962, T. Šalát [7] verified that C_λ is not even an S-set for $\lambda > 1/3$. We prove the following result.

THEOREM I. The set C_λ is a CS-set for each $\lambda \leq 1/3$.

Proof. Choose $\lambda \leq 1/3$, and set $D_n = D(A_n)$. Since $\{A_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of compact sets, it follows (see [3]) that

$$D(C_\lambda) = D(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} D_n \quad .$$

In order to verify that C_λ is a CS-set, it suffices to show that $D_n = [0,1]$ for each index n. This will be accomplished by mathematical induction. Since $\lambda \leq 1/3$, we have

$$D_1 = [0,(1-\lambda)/2] \cup [\lambda,1] = [0,1] \quad .$$

Now assume that $D_{n-1} = [0,1]$, and let d be an arbitrary element of $[0,1]$. We now show that $d \in D_n$.

From the pairs of points in A_{n-1} whose difference is d, we select a pair of points x and y such that one of them, say x, is the left endpoint of some closed interval $A_{(n-1)i}$. Thus, x is in A_n. If y is also in A_n, then clearly $d \in D_n$. Suppose that $y \notin A_n$. Then, for some index k, the point y lies between A_{nk} and $A_{n(k+1)}$. Since the distance between A_{nk} and $A_{n(k+1)}$ equals λs_{n-1} which is less than or equal to s_n, it follows that $y + \lambda s_{n-1}$ and $x + \lambda s_{n-1}$ are in A_n; hence $d \in D_n$. ∎

§3. The set T_λ.

In 1955, Bose Majumder and Sen Gupta [1] verified that T_λ is a CS-set for $\lambda = 1/(2n+1)$ [n=1,2,...]; however, as a corollary to Theorem I, we have that T_λ is a CS-set for $\lambda \le 1/3$. In 1961, Bose Majumder [2] proved that T_λ is not even an S-set for $\lambda > \sqrt{2} - 1$. Recently, the authors [4] filled the gap by proving the following result.

THEOREM II. *The set* T_λ *is a* CS-*set for* $1/3 < \lambda \le \sqrt{2} - 1$.

Remark. N. C. Bose Majumder [3] presented the following theorem: *If* $A_1, A_2, \ldots, A_n$ *are* n *symmetric sets in* [0,1], *then the set* $A_1 \times A_2 \times \cdots \times A_n$ *is a* CS-*set if, and only if, each* A_i (i=1,2,...,n) *is a* CS-*set*. This implies that T_λ is a CS-set if, and only if, $\lambda \le 1/3$, which in view of our theorem is clearly false. The invalidity of Bose Majumder's result can be seen by a more elementary example: for n=2, take $A_1 = [0,1/4] \cup [3/4,1]$ and $A_2 = [0,1]$.

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