Real Analysis Exchange Vol. 2 (1976)

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On Approximate Schwarz Derivates

Zygmund's characterization of montone functions and Dini's theorem are well-known [8]. These results are proved to be valid for Schwarz derivates and approximate derivates under suitable conditions [1, 5, 2, 6, 4]. A natural question whether they hold for approximate Schwarz derivates arises. Effort was made in [3], unfortunately the proof was in doubt (MR 47 #2010). Motivated by the fact that the concept of a strict maximum is often used when one works with Schwarz derivates (for example, [1] and [2]), we define the concept of an approximate maximum as follows:

f assumes an approximate maximum at x if the set $\{\xi : f(\xi) < f(x)\}$ has x as a point of density.

With the aid of an interesting density property proved by O'Malley [4], the following theorems are proved in [7].

Theorem 1. Let f be a Baire-1 function with

$$(*) \qquad \limsup_{h\to 0+} \text{ap } f(x-h) \leq f(x) \leq \limsup_{h\to 0+} \text{ap } f(x+h)$$

for every x. If $f(F) \cup K$ contains no non-degenerated interval, then f is non-decreasing, where F is the set of points x at which the lower approximate Schwarz derivate $\underline{f}^{(s)}_{ap}(x) \leq 0$, and K is the set $\{\alpha \in R : \{x : f(x) = \alpha$ and f assumes an approximate maximum at $x\}$ is uncountable$\}$.

Corollary. Let f be a Baire-1 function with property (*) and satisfy the condition T_3, that is, there is a dense subset R' of the real line R such that for every $c \in R'$ the set $\{\alpha \in R : \{x : f(x) = \alpha - cx\}$ is uncountable$\}$ is of Lebesgue measure zero. If $\underline{f}^{(s)}_{ap} \geq 0$ nearly everywhere, then f is non-decreasing.

Theorem 2. If f is approximately continuous and satisfies the condition T_3, then

$$\sup \left\{ \bar{f}^{(s)}_{ap}(x) : x \in R \right\} = \sup \left\{ \frac{f(x) - f(y)}{x - y} : x \neq y \right\}$$

and

$$\inf \left\{ \underline{f}^{(s)}_{ap}(x) : x \in R \right\} = \inf \left\{ \frac{f(x) - f(y)}{x - y} : x \neq y \right\},$$

where $\bar{f}^{(s)}_{ap}(x)$ denotes the upper approximate Schwarz derivate of f at x.

Theorem 3. If f is approximately continuous, satisfies the condition T_3 and the set of points where the left (right) upper and lower approximate derivates are different is countable, then

$$\sup\{\bar{f}^{(\prime)}_{ap}(x) : x \in R\} = \sup\{\underline{f}_{ap}(x) : x \in R\} = u$$

and

$$\inf\{\bar{f}^{(\prime)}_{ap}(x) : x \in R\} = \inf\{\underline{f}_{ap}(x) : x \in R\} = \ell,$$

where

$$u = \sup\left\{\frac{f(x) - f(y)}{x - y} : x \neq y\right\}, \quad \ell = \inf\left\{\frac{f(x) - f(y)}{x - y} : x \neq y\right\}.$$

We conclude this paper with the following remarks.

(1) If f is measurable, it can be shown that the set $E = \{x : f$ assumes an approximate maximum at $x\}$ is of measure zero. A question whether the image $f(E)$ is a set of measure zero in case f is approximately continuous remains open. If the answer is affirmative, then the condition T_3 in the above results can be removed.

(2) O'Malley posed a question:

Let $A \subset (x_1, x_2)$ be nonempty such that A has left density 1 at every $x \in A$ but is not necessarily an F_σ set. Does there exist an $x_0 \in [x_1, x_2) - A$ which is a right point of density of A?

If the answer is yes, then the hypothesis about the condition T_3 can be omitted too.

References

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Received September 29, 1976

Editor's note: The question asked in remark (1) of this paper has recently been answered in the negative by James Foran in his paper <u>On the density maxima of a function</u>, which is to appear in Colloq. Math.