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An Observation on Neugebauer's Characterization of Derivatives

Recently, A. M. Bruckner in the interesting article [1] has mentioned the characterization of derivatives by C. J. Neugebauer [4], and pointed out that the problem of characterizing certain types of generalized derivatives (e.g. Dini derivatives, approximate derivatives, Peano derivatives and symmetric derivatives) appears to be unsolved (and difficult!). Neugebauer's characterization is as follows:

Theorem A. Let f be a real-valued function defined on the unit interval $I_0 = [0,1]$. Then f is a derivative if and only if for each non-degenerate compact interval $I \subseteq I_0$ there exists a point x_I in the interior of I such that the following two conditions hold simultaneously:

(1) For each $x \in I_0$, $I \longrightarrow x$ implies that $f(x_I) \longrightarrow f(x)$.

(2) The interval function $f(x_I)|I|$ is additive (i.e. for any two abutting intervals $I, J \subseteq I_0$,

$$f(x_{I \cup J})|I \cup J| = f(x_I)|I| + f(x_J)|J|).$$

Here $|I|$ denotes the Lebesgue measure of I and

$I \longrightarrow x$ means that $x \in I$ and $|I| \longrightarrow 0$.

Before giving this characterization for derivatives, Neugebauer has first proved that the existence of x_I's to satisfy the condition (1) alone characterizes the class of Baire one and Darboux functions on I_0. Hence it is used to be thought that the condition (2) picks out the derivatives from the Baire one and Darboux functions. However, there are functions f which are not derivatives and the condition (2) still holds for certain choices of x_I's. For example, if f is an approximate derivative or a Peano derivative or even an approximate Peano derivative, then (2) holds for certain choices of x_I's since each of these generalized derivatives has a mean value theorem similar to the one for the ordinary derivative (see [2] for approximate derivatives, [5] for Peano derivatives, and [3] for approximate Peano derivatives). Therefore, it is more suitable to look at theorem A that it is precisely the condition (1) which picks out derivatives from those functions f satisfying the condition (2). With this observation, we obtain easily a characterization for the approximate derivatives by somehow weakening the condition (1).

Theorem 1. Let f be a real-valued function defined on the unit interval $I_0 = [0,1]$. Then f is an approximate derivative if and only if for each nondegenerate compact interval $I \subseteq I_0$ there exists a point x_I in the interior of I such that $(1)_{ap}$ and (2) hold simultaneously, where $(1)_{ap}$ is as follows:

$(1)_{ap}$. For each $x \in I_0$ there exists a closed set E(x) having density 1 (relative to I_0) at x such that $I \longrightarrow x$ with end points of I in E(x) implies $f(x_I) \longrightarrow f(x)$.

It is interesting to note that the existence of x_I's to satisfy the condition $(1)_{ap}$, which is weaker than the condition (1), also characterizes the Baire one and Darboux functions (cf. [6]). Whether it is possible to obtain a suitable condition which is weaker than the condition (1), which alone still characterizes the Baire one and Darboux functions, and which together with the condition (2) would characterize the Peano derivatives or the approximate Peano derivatives requires further investigation.

References

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