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THE LEVEL STRUCTURE OF TYPICAL CONTINUOUS FUNCTIONS

The purpose of this note is to describe the behavior of a "typical" continuous function defined on the interval [0,1] in terms of its level set structure. By "typical" we mean that all continuous functions, except for those in some first category subset of C = C[0,1], exhibit the behavior we describe, and by "level set structure" of a continuous function f we mean the manner in which the graph of f intersects straight lines. Although a typical continuous function exhibits behavior which is in some sense pathological, we shall see that, viewed in terms of levels, the behavior is very regular. Every typical function behaves very much like every other typical function. Viewed in terms of this behavior, a number of pathological properties become more plausible. Indeed, some of these pathological properties follow (without too much effort) from our results.

In 1939 Gillis [2] constructed a continuous function f with the property that for every real number c, the set $\{x: f(x) = c\}$ is perfect. The problem of determining whether such behavior is typical was raised in [1]. The following theorem shows this problem has a negative answer.

THEOREM A. A typical function in C has its "top" and "bottom" horizontal levels singletons and all levels between the top and bottom uncountable. Of these, all but countably many are perfect: the remaining levels consist of a perfect set together with a single isolated point. These levels with a single isolated point correspond to a dense set of heights between the maximum and minimum values assumed by the function.

There is, of course, no special role played by the horizontal direction. For any fixed real number a, the intersection of the graph of a typical continuous function with the family of lines $\{ax+b\}$ exhibits a similar behavior.

The Gillis function g actually has the property that the intersection of the graph of g with <u>every</u> non-vertical line is perfect; thus, his function exhibits the same behavior in all non-vertical directions as it does in the horizontal direction. Theorem B shows that a typical continuous function has the kind of behavior described in Theorem A for all but countably many directions.

THEOREM B. A typical continuous function possesses the behavior described in Theorem A in all but a countable dense set of directions. For these exceptional directions the behavior is the same except for exactly one line in each such direction. These exceptional lines in the exceptional directions intersect the graph in a perfect set P together with exactly <u>two</u> isolated points. (In case the exceptional line is an extreme one, the set P is empty).

The exceptional lines and exceptional directions depend on the function, of course.

These theorems, along with certain lemmas and preliminary theorems, can be used to prove a number of known theorems concerning differentiation properties of typical continuous functions. Note, for example, that if a line $cx + b$ intersects the graph of f in a nonempty perfect set, then the number c must be a derived number of f at each point corresponding to a point of intersection. Thus, Theorem B makes more plausible the following theorem of Jarnik [3]: A typical continuous function has every extended real number as a derived number at every point. We base an actual proof of Jarnik's theorem on certain results we obtain prior to establishing Theorems A and B above.

Theorem B suggests certain questions which we have not answered.

(1) Suppose we replace the family $\{ax + b\}$ of lines by some other two parameter family $\mathfrak{F}$: i.e. $\mathfrak{F}$ is a family of functions with the property that if x_1, x_2, y_1, y_2 are real numbers, $x_1 \neq x_2$, there exists a unique $f \in \mathfrak{F}$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. What conditions on $\mathfrak{F}$ will guarantee that the analogues of Theorem B hold with $\mathfrak{F}$ replacing the family $\{ax+b\}$? e.g. is it sufficient for each $f \in \mathfrak{F}$ to satisfy a Lipschitz condition?

(2) Let P_n denote the family of polynomials of degree at most n. What can one say about the intersections of a typical continuous f

with the members of P_n? For example, is it true that the intersection of a typical f with $p \in P_n$ contains at most n + 1 isolated points? And must there exist a $p \in P_n$ whose intersection with f does have n + 1 isolated points. In general, if Q is a subset of P_n obtained by fixing k specified coefficients and allowing the remaining coefficients to be arbitrary, then what intersection properties does a typical f have with the members of Q?

(3) If C [0,1] is replaced with some other complete metric space of functions, (e.g. the set of bounded Darboux Baire 1) functions furnished with the "sup metric", what can now be said about intersections of typical functions in the class with straight lines?

(4) If one replaces C [0,1] with the family of continuous functions defined on a square, what are the intersections of typical functions with straight lines or with planes? Information here might yield results about differentiability properties of typical continuous functions of several variables.

References

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3. V. Jarnik, Uber die Differenzierbarkeit stetiger Funktionen, Fund. Math. 21 (1933), 48-58.

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