MULTIPLICATION AND THE FUNDAMENTAL THEOREM OF CALCULUS - A SURVEY

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Because the limit process behaves nicely when combined with the arithmetic operations, many classes of functions one encounters in analysis are closed under multiplication. The integrals of Newton, Riemann, Lebesgue and Denjoy, for example, have this property. However, the corresponding classes of integrands do not.

The problem of determining conditions under which the product of two derivatives is a derivative was first attacked in 1910 by W. H. Young [24], so it seems fitting to begin this survey with the following excerpt from his introduction.

> Recent research has provided us with a set of necessary and sufficient conditions that a function may be an indefinite integral, in the generalised sense, of another function, and the way has thus been opened to important developments. The corresponding, much more difficult, problem of determining necessary and sufficient conditions that a function may be a differential coefficient, has barely been mooted; indeed, though we know a number of necessary conditions, no set even of sufficient conditions has to my knowledge ever been formulated, except that involved in the obvious statement that a continuous function is a differential coefficient. The necessary conditions in question are of considerable importance and interest. A function which is a differential coefficient has, in fact, various striking properties. It must be pointwise discontinuous with respect to every perfect set; it can have no discontinuity of the first kind; it assumes in every interval all values between its upper and lower bounds in that interval; its value at any point is one of the limits on both sides of the values in the neighbourhood; its upper and lower bounds, when finite, are unaltered if we omit any set of points of con-

tent zero, and, in the general case, are unaltered, if we omit the values at any countable set of points; the points at which it is infinite form an inner limiting set of content zero. From these necessary conditions we are able to deduce much valuable information as to when a function is certainly not a differential coefficient. They enable us to realise the very special characteristics of a function which is a differential coefficient. It is clear that, for example, a function which is a differential coefficient ceases to be a differential coefficient if its value be altered at a single point. These conditions do not, however, render us any material assistance, even in answering the simple question as to whether the product of two differential coefficients is a differential coefficient, and this not even in the special case in which one of the differential coefficients is a continuous function.

It is with this last-named case that the present paper is concerned. In view of the importance of the problem in the theory of the differentiation of infinite series and of improper integrals - where the property of being a differential coefficient presents itself naturally as the necessary and sufficient condition which must hold in order that such differentiation may, on certain assumptions, be permissible - as well as in other applications, it is hoped that the results here obtained will be regarded as of interest.

The purpose of this survey is to give as complete an account as possible of the progress that has been made on the problem of products of derivatives since Young's paper. The results and examples are grouped in what, it is hoped, the reader will find to be a reasonable mathematical order. Although the chronological development of the subject suffers somewhat from this, it is easily recaptured from publication dates.

For an account of the progress made on the more important problem raised in Young's introduction, that of characterizing the class of derivatives, the reader is referred to [1] and [2].

§1. Preliminary Remarks

Throughout this survey all functions will be defined on [0,1] and will assume finite real values unless otherwise noted.

The problem of determining conditions under which the product of two functions, f and g, which belong to a class Q is again a member of Q can be approached in two ways. The first is to find conditions on f alone which will insure that its product with each member of Q also belongs to Q.

Definition 1.1. *For a class of functions* Q, *its multiplier class* M(Q) *is the set of all functions* f *such that* $g \in Q$ *implies* $fg \in Q$.

It is clear that M(Q) is always closed under multiplication, that $1 \in Q$ implies that $M(Q) \subseteq Q$ and that if Q is closed under addition, so is M(Q).

The second approach is to place less restrictive conditions on each of the functions. The first method does have the advantage that M(Q) can often be characterized, whereas the second yields only sufficient conditions.

It was noted in the introduction that the integrals of Newton, Riemann, Lebesgue and Denjoy are closed under multiplication. For the Newton integral (antidifferentiation) this is just the product rule. For the others it suffices to note that the classes of absolutely continuous (AC), generalized absolutely continuous in the restricted sense (ACG_*) and generalized absolutely continuous (ACG) functions are closed under multiplication. (A reader unfamiliar with the restricted sense Denjoy integral (D_*) and the wide sense

Denjoy integral (D) is referred to [18, Chs. VII and VIII].) Then if F is the integral of f and G is the integral of g, one can easily show that $(Fg + fG)$ is integrable in the appropriate sense and that FG is its integral.

The fundamental theorem states that the derivative of a Riemann, Lebesgue or a restricted sense Denjoy integral exists and equals the integrand almost everywhere (and that the same is true for the approximate derivative of a wide sense Denjoy integral). This raises the question of products of functions which are almost everywhere the derivative of a continuous function. However, a theorem of Lusin [18, p.217] asserts that every measurable function is almost everywhere the derivative of a continuous function. Thus this class is closed under multiplication and we shall restrict our attention to functions which are derivatives at each point of $[0,1]$. Consequently, the fundamental theorem takes the following form for the purposes of this survey.

Theorem 1.1. *A function is a summable derivative if and only if it is the derivative of its Lebesgue integral.*

Theorem 1.2. *A function is a derivative if and only if it is the derivative of its restricted sense Denjoy integral.*

Theorem 1.3. *A function is the approximate derivative of a continuous function if and only if it is the approximate derivative of its wide sense Denjoy integral.*

We conclude this section with a slight modification of an example given in [7] which shows the need for the parentheses in the expression (Fg + fG) when asserting that it is integrable and which points out the pathological behavior of products of derivatives.

Example 1.1. *There exist functions F and G such that F and G are everywhere differentiable (and therefore D_* integrable) such that FG' and F'G fail to be integrable (in any of the senses under consideration) and are, therefore, not derivatives.*

Construction. For $x \in (0,1]$, we define the functions

$$F(x) = x^2 \sin(x^{-5}); \quad g(x) = x^{-4} \sin(x^{-5}) \quad ;$$

$$G(x) = (1/5)x^2 \cos(x^{-5}) - (2/5) \int_0^x t \cos(t^{-5})dt \quad ;$$

and

$$H(x) = (1/10)x^4 \sin(x^{-5})\cos(x^{-5}) + 1/2x^2$$

$$- (2/5) \int_0^x t^3 \sin(t^{-5})\cos(t^{-5})dt \quad .$$

Set $F(0) = G(0) = g(0) = 0$. It is easily verified that F is differentiable, that $G'(x) = g(x)$ on $[0,1]$ and that $H'(x) = F(x)g(x)$ on $(0,1]$. Since $\lim_{x\to 0} H(x) = +\infty$, $F(x)g(x)$ is not integrable in any of the above senses. But since $(Fg + F'G)$ is a derivative by the product rule, it is D_* integrable and it follows by linearity that F'G is not integrable.

We conclude this section by noting that Theorem 4.4 shows that neither F nor G can have summable derivatives.

§2. Products of Integrable Functions

Let R denote the class of Riemann integrable functions on [0,1]. Then f belongs to R if and only if it is bounded and its discontinuities form a set of measure 0. Since these properties are closed under multiplication, we have

<u>Theorem 2.1</u>. M(R) = R.

The following characterization of the multipliers for the class of summable functions L, important in its connection to the fact that L_∞ is isomorphic to the dual space of L_1, is due to Lebesgue [16].

<u>Theorem 2.2</u>. <u>The class</u> M(L) <u>is the set of measurable and essentially bounded functions</u>.

In 1948 W.L.C. Sargent [19] gave the following description of the multiplier class for the restricted sense Denjoy integrable functions D_* and the wide sense Denjoy integrable functions D. This result provides a starting point for the investigation of the multipliers of derivatives and approximate derivatives of continuous functions not only in its connection with the fundamental theorem, but also because it indicates the importance of the variation of a multiplier.

From the integration by parts formula for D_* and D [18, p.246], one sees that if F is of bounded variation or agrees almost everywhere with a function of bounded variation, then F belongs to $M(D_*)$ and M(D).

Sargent defines the essential oscillation and essential variation of a function F on an interval and shows that F agrees almost everywhere with a bounded variation function if and only if the essential variation of F is finite. Assuming the essential variation to be infinite, a function $g \in D_* \subset D$ is constructed such that Fg is not in D_* nor in D.

Theorem 2.3. $M(D_*) = M(D) = \{F \mid F$ *is of essential bounded variation*$\}$.

One can also establish this theorem by considering the function $\hat{F}(x) = \lim_{t \to x^+} \text{ap } F(t)$. It follows from Theorem 2.2 that F is measurable and essentially bounded. Thus $\hat{F} = F$ almost everywhere and one can construct the above mentioned counterexample g if this limit fails to exist at some point $x \in [0,1)$ or if the function $\hat{F}$ fails to be of bounded variation [6, p.12].

We conclude this section by noting that it is possible to construct unbounded derivatives whose product with every derivative is in D_* [6, p.29].

§3. Products of Bounded Derivatives

Let BD denote the class of bounded derivatives on the interval [0,1]. Young [24, Theorem 1] established the following sufficient condition for the product of two members of BD to belong to BD.

Theorem 3.1. *If* f *belongs to* BD *and* g *is continuous, then* fg *belongs to* BD.

A more general sufficient condition and the characterization of the multiplier class M(BD) were obtained in 1921 by W. Wilkosz [22].

Theorem 3.2. *If f and g belong to BD and if at each point x at least one of them is upper (or lower) semicontinuous, then fg belongs to BD.*

Theorem 3.3. *For f and f^2 to belong to BD, it is necessary and sufficient that f be a bounded approximately continuous function.*

Wilkosz then attributes the proof of the following result to Stephan Banach.

Theorem 3.4. *If f, f^2 and g belong to BD, then fg belongs to BD.*

The previous results actually characterize the multiplier class M(BD) since $f \in M(BD)$ implies $f \cdot 1$ and then $f \cdot f$ are bounded derivatives.

Theorem 3.5. *A function f belongs to M(BD) if and only if f and f^2 belong to BD.*

Theorem 3.6. *A function f belongs to M(BD) if and only if f is bounded and approximately continuous.*

This last result was also established by J. Wolff [23] and M. Iosifescu [13]. Iosifescu obtains further descriptions of M(BD) by noting that the notions of approximate continuity, Lebesgue point of the first kind

$$\lim_{h\to 0}(1/h)\int_x^{x+h} |f(t) - f(x)|dt = 0$$

and Lebesgue point of the second kind

$$\lim_{h\to 0}(1/h)\int_x^{x+h} (f(t) = f(x))^2 dt = 0$$

are equivalent for bounded measurable functions.

The various proofs of Theorem 3.6 involve fixing a point x and showing that because f is approximately continuous at x, fg is the derivative of its Lebesgue integral at x. Thus one obtains the following sufficient condition for the product of two members of BD to belong to BD. This theorem is given a direct proof in [13].

Theorem 3.7. If f *and* g *are bounded derivatives and if at each point of* $[0,1]$ *at least one of them is approximately continuous, then* fg *belongs to* BD.

Iosifescu notes that approximate continuity is not a necessary condition since if

$$f(x) = \sin(1/x) \quad \text{and} \quad g(x) = \cos(1/x)$$

with $f(0) = g(0) = 0$, then f, g, and fg are derivatives but neither f nor g are approximately continuous at $x=0$. He also notes that since a bounded derivative cannot be upper or lower semicontinuous at a point without being approximately continuous there [15], Theorem 3.7 implies Theorem 3.2.

We conclude this section by noting that f and g in the above example "look alike" and yet fg is a derivative whereas ff and gg are not. Thus a definitive solution to the multiplication problem for bounded derivatives by placing conditions weaker than approximate continuity on each of the two functions seems unlikely unless, perhaps, a characterization for the class of bounded derivatives itself can be found.

§4. Products of Summable Derivatives

Let SD denote the class of summable derivatives on [0,1]. It is of interest to note that the problem of characterizing M(SD) remains unsolved while the seemingly more difficult task of characterizing the multiplier class for all (finite) derivatives is complete [Theorem 5.5].

Problem 4.1. *Characterize* M(SD).

Throughout most of his paper, Young assumes the summability of the derivatives with which he is working. This is primarily due to the fact that the D_* integral, the necessary tool for working with derivatives, was not defined until 1912 [5]. We shall see in Section 5 that the assumption that f is summable is unnecessary in the following result [24, Theorem 2].

Theorem 4.1. *If* f *belongs to* SD *and* g *is a continuous function of bounded variation, then* fg *belongs to* SD.

If a derivative is bounded above or below, then it is a summable derivative [18, p.242]. Also if f and f^2 are derivatives, then both are summable since $f^2 \geq 0$ and $|f| \leq \max\{f^2,1\}$.

Young obtained the following theorem as a corollary to [24, Theorem 5] which, as we shall see in Section 7, is not correct.

Thus a direct proof is supplied.

<u>Theorem 4.2. If f is a derivative with an upper or lower bound and g is continuous, then fg belongs to SD.</u>

<u>Proof</u>. We may assume that $f(x) \ge 0$ and $g(x) \ge 0$. Fix x in $[0,1]$ and set $\varepsilon_h = \sup\{|g(t) - g(x)| : |t - x| \le h\}$. Then $\varepsilon_h \to 0$ as $h \to 0$. Noting that

$$(1/h) \int_x^{x+h} f(t)g(t)dt \quad \text{and} \quad (1/h) \int_x^{x+h} f(t)dt$$

are positive for both positive and negative values of h, we have

$$(g(x) - \varepsilon_h)(1/h) \int_x^{x+h} f(t)dt \le (1/h) \int_x^{x+h} f(t)g(t)dt$$

$$\le (g(x) + \varepsilon_h)(1/h) \int_x^{x+h} f(t)dt \quad .$$

Since f belongs to SD,

$$\lim_{h\to 0}(1/h) \int_x^{x+h} f(t)dt = f(x) \quad ,$$

and hence $f(x)g(x)$ is the derivative of its integral.

The proof of Theorem 4.2 raises the question as to whether one could replace the continuity of g by approximate continuity and boundedness by taking approximate limits on the inequality. If so, this would show that fg is the approximate derivative of its integral and since its integral is an increasing function, the approximate derivative reduces to an ordinary derivative [11]. However, the following example shows that this is not the case.

Example 4.1. There exist derivatives f and g such that f is positive, g is bounded and approximately continuous and fg is not a derivative.

Construction. Let $I_n = [1/n+1, 1/n]$ and let $J_n = [a_n, b_n]$ be any closed subinterval of I_n such that 0 is a point of dispersion of the union of the J_n. (One can show that it suffices to choose J_n such that $|J_n| = \varepsilon_n |I_n|$ where $\varepsilon_n \to 0$.) Let K_n be a closed interval contained in the interior of J_n.

On K_n define f to be the two equal sides of an isosceles triangle whose base is K_n and whose area is equal to $|I_n|$. Set $f(x) = 0$ on $(0,1]\setminus K_n$ and set $f(0) = 1$. Since f is continuous on $(0,1]$, we need only show that it is the derivative of its integral at $x=0$ in order to show that it is a derivative. But since for all n,

$$\int_0^{1/n} f(t)dt = 1/n \quad ,$$

it follows that if x belongs to I_n, then

$$n/(n+1) \leq (1/x) \int_0^{x} f(t)dt \leq (n+1)/n$$

and $f(0) = 1$ is the derivative of the integral.

Then set $g(x) = 1$ on K_n, $g(x) = 0$ on (b_{n+1}, a_n) and at $x=0$. Define g to be linear on the remaining intervals so as to be continuous on $(0,1]$. It is easily verified that g is approximately continuous at 0 and that $f(x)g(x) = f(x)$ on $(0,1]$. Since $f(0)g(0) = 1 \cdot 0$, fg differs from a derivative at only one point and cannot be a derivative.

It follows from the construction of f that the approximate limit of f at 0 equals 0 and, therefore, f is not approximately continuous. The next theorem is an easy corollary to Theorem 7.6 which is due to Zahorski [25, p.30].

Theorem 4.3. *If f is an approximately continuous derivative which is bounded above or below and g is a bounded derivative, then fg belongs to SD.*

The following example which further illustrates the hypotheses of Theorems 4.1 and 4.2 is due to J. Wolff [23]. He comments that due to this example, "we cannot expect any progress worth mentioning in this train of thought."

Example 4.2. *For $x \in (0,1]$, define the functions*

$$f(x) = (1/\sqrt{x})\sin(1/x) \quad \text{and} \quad g(x) = \sqrt{x}\,\sin(1/x) \quad ,$$

and let $f(0) = g(0) = 0$. Then g is continuous, f is a summable derivative but fg is not a derivative.

The following result is also due to Wolff [23].

Theorem 4.4. *If f belongs to SD and g has finite derivates at each point x (g is locally Lipschitz), then fg belongs to SD.*

This theorem, which is also proved in [13], indicates why the solution to the multiplier problem for finite derivatives M(FD) does not provide a solution to M(SD). In the following section it will be shown that membership in M(FD) depends entirely on the "good" behaviour of the variation of the multiplier. However, it is easy to construct locally Lipschitz functions with "bad" variation. As a mild example, we note that $F(x) = x^2\sin(x^{-5})$ of Example 1.1 belongs to M(SD) (as do all differentiable functions), but does not belong to M(FD).

If f were an unbounded member of M(SD), then f^2 would also be an unbounded member of M(SD). Then upon examination of the proof of [8, Theorem 8], one sees that the derivative g, constructed such that fg is not a derivative, is summable.

Theorem 4.5. *Members of* M(SD) *are bounded.*

This result shows that M(SD) is contained in M(BD) since f and f^2 must be bounded derivatives. Theorem 3.6 shows that $g(x) = \sqrt{x}\sin(1/x)$ belongs to M(BD) and Example 4.2 shows that it does not belong to M(SD).

Theorem 4.6. M(SD) *is properly contained in* M(BD).

In the next section, we shall see that members of M(FD) are bounded. Since the product of a bounded function and a summable function is summable, M(SD) contains M(FD) and $F(x) = x^2\sin(x^{-5})$ shows that the containment is proper.

Theorem 4.7. M(FD) *is properly contained in* M(SD).

We conclude this section with four theorems which give sufficient conditions for the product of two members of SD to belong to SD. The first is due to Wilkosz [22] and the last three are due to Iosifescu [13].

Theorem 4.8. If f, f^2, g *and* g^2 *belong to* SD, *then* fg *belongs to* SD.

Theorem 4.9. If f *and* f^2 *are summable, then* f *and* f^2 *belong to* SD *if and only if each point* x *is a Lebesgue point of the second kind for* f.

Theorem 4.10. If f *and* g *belong to* SD, *then* fg *belongs to* SD *if each point* x *is a Lipschitz point for at least one of* f *or* g.

Theorem 4.11. Let f *belong to* BD *and* g *belong to* SD. *Then* fg *belongs to* SD *provided that each point* x *is a Lipschitz point of* f *or a Lebesgue point of the first kind for* g.

§5. Products of Finite Derivatives

Let FD denote the class of finite derivatives on the interval [0,1]. Our first result is again due to Young [24, Theorem 4].

Theorem 5.1. If f *belongs to* FD *and* g' *is bounded, then* fg *belongs to* FD.

Young's result is actually more general in that it allows f to assume the values $+\infty$ and $-\infty$. This theorem follows easily from Theorem 3.1 and the identity $fg = (Fg)' - Fg'$ where $F' = f$. Theorem 5.1 was also established by N. A. Selivanov [20] and J. C. Burkill [3]. Burkill proved it as a lemma to the integration by parts formula for the Cesaro-Perron integral.

In 1973 J. Foran [10] obtained the following theorem.

Theorem 5.2. If f *is absolutely continuous, then* f *belongs to* M(FD).

Foran's paper stimulated further research on products of derivatives which resulted in the solution of the problem in [7] and [8]. Professors M. Laczkovich and G. Petruska also obtained the characterization of M(FD) and the related examples (*On the Multipliers of Derivatives*, withdrawn). The priority of [7] and [8] is due to their author's access to Foran's paper two years before it appeared.

The following generalization of Theorem 4.1 is obtained in [7].

Theorem 5.3. If f *belongs to* FD *and* g *is a continuous function of bounded variation, then* fg *belongs to* FD.

In answer to a question raised by Foran, the following example is constructed in [7].

Example 5.1. There exists a discontinuous member of M(FD).

The function f is constructed by erecting a spike of height 1 on each of a sequence of intervals that tend to 0 and satisfy certain restrictions on their length and distance to the origin, and by setting $f(x) = 0$ at the remaining points of [0,1]. An investigation of these restrictions and the corresponding limitations they impose on the variation of a multiplier led to the following definition which, in a slightly more restrictive form, appears in [8].

Definition 5.1. A function f is said to be of distant bounded variation from the right at x_0 if there exist positive numbers M and δ such that

(i) if $0 < \alpha < \delta$, then f is of bounded variation on $(x_0+\alpha, x_0+\delta)$,

(ii) letting $dW(t)$ denote the Lebesgue-Stieltjes measure induced on $(x_0, x_0+\delta)$ by the total variation of f,

$$\int_{x_0}^{x_0+\delta} (t - x_0)dW(t)$$

exists as an improper Lebesgue-Stieltjes integral and

(iii) for each x in $(x_0, x_0+\delta)$,

$$\int_{x_0}^{x} (t - x_0)dW(t) \le M(x - x_0) \quad .$$

We abbreviate this condition by $f \in BVD^+(x_0)$. Then we write $f \in BVD^-(x_0)$ if $f(1-x) \in BVD^+(1-x_0)$; and if both conditions hold, we say that f is of distant bounded variation at x_0 and write $f \in BVD(x_0)$.

The following characterization of $M(FD)$ is then obtained.

Theorem 5.4. A function f belongs to $M(FD)$ if and only if

(a) f is a bounded derivative,

(b) there exist at most finitely many points x in every neighborhood of which f is of unbounded variation, and at such points

(c) $f \in BVD(x)$.

The conditions in Theorem 5.4 are not independent. The set of points at which a function f is not of bounded variation is closed. If this set had a limit point x_0, it is easily seen that $f \notin BVD(x_0)$ and, consequently, the assumption of finiteness in (b) is unnecessary.

The following result is established in [8].

Lemma 5.1. *If f is of bounded variation on the interval (a,b), then $f \in BVD(x)$ for each x in (a,b) and M may be chosen to be the total variation of f on (a,b).*

Thus if f belongs to $M(FD)$, then $f \in BVD(x)$ for each x in $[0,1]$. Moreover, if we choose M to be the largest of the constants corresponding to the points x at which f is of unbounded variation, then M holds for all x since we can choose δ_x so small that the total variation of f on $(x-\delta_x, x+\delta_x)$ is less than M. We note, however, that if f is not of bounded variation, then δ_x cannot be chosen independently of x for $(x-\delta_x, x+\delta_x)$ must exclude any point at which f is of unbounded variation.

Lemma 5.2. *If f is a derivative and $f \in BVD(x)$, then f is bounded in a neighborhood of x.*

Proof. We may assume that $x=0$, that $f(0) = 0$ and that $f \in BVD^+(0)$. Then since f is a derivative,

$$\lim_{x \to 0} (1/x) \int_0^x f(t)dt = f(0) = 0 \quad .$$

Thus we may choose $\delta > 0$ such that if $0 < x < \delta$,

$$\left|(1/x)\int_0^x f(t)dt\right| < 1$$

and

$$(1/x)\int_0^x t\,dW(t) < M \quad \text{for some } M > 1 \quad .$$

If f is unbounded at 0, we may assume that $\overline{\lim_{x\to 0}}\, f(x) = +\infty$. Choose $b < \delta$ such that $f(b) = K > 8M$.

Case 1) $f(x) \geq K/2$ on $[b/2,b]$. Then

$$(1/b)\int_0^b f(t)dt = (1/2)(2/b)\int_0^{b/2} f(t)dt + 1/b\int_{b/2}^b f(t)dt$$

$$\geq -1/2 + (1/b)(b/2)(K/2) > 2M - (1/2) > M \quad .$$

Case 2) $f(x) < K/2$ for some x in $[b/2,b]$. Then the total variation of f on $[b/2,b]$ is greater than $K/2$, that is,

$$\int_{b/2}^b dW(t) > K/2 \quad .$$

Then

$$(1/b)\int_0^b t\,dW(t) \geq (1/b)\int_{b/2}^b (b/2)dW(t)$$

$$\geq (1/2)(K/2) > 2M$$

Thus a contradiction is obtained in each case and the lemma is established.

Thus if f is a derivative and $f \in BVD(x)$ for each x in $[0,1]$, then f is bounded on $[0,1]$ by the Heine-Borel Theorem.

These considerations show that Theorem 5.4 can be improved as follows.

Theorem 5.5. *A function* f *belongs to* $M(FD)$ *if and only if* f *is a derivative and* $f \in BVD(x)$ *for each* x *in* $[0,1]$.

Before showing the independence of these two conditions, we note that the following theorem is an easy consequence of Theorems 5.4 and 3.6 and Example 1.1.

Theorem 5.6. $M(FD)$ *is properly contained in* $M(BD)$. *Therefore members of* $M(FD)$ *are approximately continuous.*

The function $F(x) = x^2\sin(x^{-5})$ of Example 1.1 is a derivative but $F \notin BVD^+(0)$. If it were, it would belong to $M(FD)$ by Theorem 5.5 since it is of bounded variation elsewhere on $[0,1]$.

Example 5.2. *There exists a bounded function* f *such that* $f \in BVD(x)$ *for each* x *but* f *is not a derivative.*

Construction. Let $I_n = [a_n, b_n] = [2^{-n}, 2^{-n}+4^{-n}]$, $J_n = [b_{n+1}, a_n]$ if n is even and $K_n = [b_{n+1}, a_n]$ if n is odd. Set $f(x) = 1$ on $J = \bigcup_n J_n$, $f(x) = 0$ on $K = \bigcup_n K_n$ and define $f(x)$ to be linear on I_n so that it is continuous on $(0,1]$. We leave $f(0)$ undefined. We note that the variation of f on J_n and K_n is 0 and on I_n it is 1. Thus,

$$\int_{a_n}^{b_n} t\,dW(t) < \int_{a_n}^{b_n} (2^{-n}+4^{-n})\,dW(t) = 2^{-n}+4^{-n} \quad .$$

Then if $a_{n+1} \le x \le a_n$,

$$(1/x)\int_0^x t\,dW(t) \le (1/a_{n+1})\int_0^{a_n} t\,dW(t)$$

$$\le 2^{n+1}\sum_{K=n+1}^{\infty}(2^{-K}+4^{-K}) \le 3 \quad .$$

Thus $f \in BVD^{+}(0)$ and since f is of bounded variation on closed intervals which do not contain 0, $f \in BVD(x)$ for each x in $[0,1]$.

It is easily verified that

$$\overline{\lim_{h\to 0}}\ |[0,h]\cap J|/h = \overline{\lim_{h\to 0}}\ |[0,h]\cap K|/h \ge 1/2 \quad .$$

Therefore, there is no value $f(0)$ for which f is approximately continuous on $[0,1]$. By Theorem 5.6, $f(x)$ cannot belong to $M(FD)$ for any choice of $f(0)$. By Theorem 5.5, $f(x)$ cannot be a derivative for any choice of $f(0)$.

In the preceding section it was noted that there is a major difference between $M(SD)$ and $M(FD)$ due to the fact that locally Lipschitz functions can have variations which behave badly. We conclude this section with another example of this.

Example 5.3. There exists a function f which belongs to $M(SD)$ such that the set of points x at which f is not of distant bounded variation is of positive measure.

Construction. Let P be a nowhere dense, perfect set of positive measure contained in $[0,1]$. Then let $I_n = (a_n,b_n)$ be the sequence of intervals contiguous to P. On I_n erect the equilateral triangle whose base is I_n and which lies above the x-axis. Define f on I_n to be a piecewise linear, continuous function such that f is of bounded variation on intervals of the form $(a_n+\delta, b_n-\delta)$, the graph of f lies inside the triangle on I_n, and $f \notin BVD^+(a_n)$. Set $f(x) = 0$ on P. Then since the graph of f lies within the triangles on the I_n, f is locally Lipschitz and belongs to $M(SD)$. Since the set $\{a_n\}$ is dense in P and since the set of points at which f is not BVD is closed, we have that $f \notin BVD(x)$ on P and the example is complete.

§6. Products of Approximate Derivatives

Let ADC denote the class of functions which are the approximate derivative of a continuous function on the interval $[0,1]$ and let AD denote the class of functions which are approximate derivatives on $[0,1]$.

The following characterizations of the multiplier classes are contained in [9]. (The proofs are sketched in the first issue of the _Real Analysis Exchange_.)

Theorem 6.1. A function f _belongs to_ $M(ADC)$ _if and only if_ f _is of bounded variation and its total variation is locally Lipschitz._

Theorem 6.2. A function f _belongs to_ $M(AD)$ _if and only if_ f _is constant on_ $[0,1]$.

§7. Extended Real Valued Derivatives

Let EVD denote the class of extended real valued functions f such that f is the derivative of a continuous function at each point x of $[0,1]$.

The first problem one encounters with the class EVD is that it is not closed under addition since the sum $(+\infty) + (-\infty)$ is undefined and in order to make any claims concerning the difference of two members of EVD, one must be certain that both do not assume an infinite value (with the same sign) at any point.

A more serious difficulty is that the fundamental theorem breaks down in the following sense; if $F'(x)$ is summable, its indefinite integral need not differ from F by a constant. (However, if F' is summable, it is also the derivative of its indefinite integral.) An example of such a function is constructed in [18, p.205]. We circumvent this difficulty with the following convention: the statement "F' is summable (or D-integrable)" entails that F be its Lebesgue (or Denjoy) integral.

Young [24] showed that under certain restrictions, the product rule still holds.

Theorem 7.1. *If* F *and* G *are differentiable in the extended sense, then* $(FG)' = F'G + FG'$ *provided that* (i) F' *and* G' *do not assume infinite values at the same point and* (ii) *the indeterminate forms* $(+\infty)\cdot 0$ *and* $(-\infty)\cdot 0$ *are defined to be* 0.

The next two theorems are also due to Young.

Theorem 7.2. *If* $f \in EVD$ *is the derivative of a continuous function of bounded variation* F, *and* g' *is finite and summable, then* fg *belongs to* EVD.

This theorem follows from Theorem 7.1 and the fact that Fg'

is a finite derivative by Theorem 4.1. Theorem 5.3 shows that the assumption that g is summable is unnecessary.

Theorem 7.3. *If f belongs to EVD and g' is bounded, then fg belongs to EVD.*

This follows immediately from Theorem 7.1 and Theorem 3.1.

Young's next result [24, Theorem 5] is incorrectly stated (unless we take it as a blanket assumption that the two functions do not assume infinite values at the same point). He asserts that if f is a summable member of EVD which is bounded above or below and g is continuous, then fg belongs to EVD. The next example shows that this is not correct.

Example 7.1. *There exists a positive, summable function f which belongs to EVD and an absolutely continuous function g such that fg does not belong to EVD.*

Construction. For x in the interval $[0,1/2)$ set

$$f(x) = 1/\sqrt{-x + 1/2}\ , \quad g(x) = -2\sqrt{-x + 1/2}\ , \quad F(x) = -2\sqrt{-x + 1/2}\ .$$

For x in $(1/2,1]$, define

$$f(x) = 1/\sqrt{x - 1/2}\ , \quad g(x) = \sqrt{x - 1/2}\ , \quad F(x) = 2\sqrt{x - 1/2}\ .$$

Set $F(1/2) = g(1/2) = 0$ and $f(1/2) = +\infty$. Then F and g are absolutely continuous and $F'(x) = f(x)$ on $[0,1]$. Since $f(x)g(x) = -2$ on $[0,1/2)$ and $f(x)g(x) = 1$ on $(1/2,1]$, no matter what value one assigns to $f(1/2)g(1/2)$, $fg \notin \text{EVD}$.

Our next objective is to state and prove an amended version of Young's theorem.

Theorem 7.4. *If* f *is a summable member of* EVD *which is bounded below (or above) and* g *is a continuous function, then* fg *belongs to* EVD *provided that* g *satisfies a Lipschitz condition at each point* x *for which* $f(x) = +\infty$ $(f(x) = -\infty)$ *and* $g(x) = 0$.

Proof. We assume f is bounded below by A. Since $f(x)g(x)$ belongs to EVD if and only if $(f(x) - A)g(x)$ belongs to EVD (because $Ag(x)$ is a finite derivative), we may suppose that $f(x) \geq 0$. (We note that this theorem and the previous example show that one cannot remove a zero of g by considering $f(x)(g(x) + B)$ which is done in [24].)

Noting that fg is summable, we see that the proof of Theorem 4.2 shows that fg is the derivative of its integral at each point at which f is finite or g is not 0. Thus if $f(x) = +\infty$ and $g(x) = 0$, we must show that $f(x)g(x) = (+\infty)\cdot 0 = 0$ is the derivative of its integral. Since g satisfies a Lipschitz condition at x,

$$|[g(t) - g(x)]/(t-x)| = |g(t)/(t-x)| \leq N \quad .$$

Let $\varepsilon_h = \sup\{|g(t)|: |t - x| \leq |h|\}$. Then since

$$|g(t)/h| \leq |g(t)/(t-x)| \leq N \quad \text{for} \quad 0 < |t - x| \leq |h| \quad ,$$

we have $|\varepsilon_h/h| \leq N$, and since $f(x) \geq 0$,

$$\left|(1/h)\int_x^{x+h} f(t)g(t)dt\right| \leq \left|(1/h)\varepsilon_h \int_x^{x+h} f(t)dt\right|$$

$$\leq N\cdot\left|\int_x^{x+h} f(t)dt\right| \quad .$$

Since f is summable, this last term tends to 0 as h tends to 0 and Theorem 7.4 is established.

The following result is a corrected form of Theorem 6 in [24].

Theorem 7.5. *If f belongs to EVD and g' is finite and bounded above or below, then fg belongs to EVD.*

This result follows from Theorem 7.1 and the fact that fg' is a finite derivative by Theorem 4.2.

It was noted in the introduction that Young's paper raises the more important problem of finding characterizations of the various classes of derivatives. Although the problem remains unsolved, it seems appropriate to end this survey with a result from the 1950 paper by Z. Zahorski [25, p.30] which, in the opinion of this author, remains the most important paper on this problem.

Theorem 7.6. *If f is an approximately continuous derivative such that $0 \le f(x)$, g is a derivative such that $0 \le g(x) \le 1$, and $\{x|f(x)=+\infty\} \cap \{x|g(x)=0\}$ is void, then fg belongs to EVD.*

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* These papers are of related interest, but are not cited in the text.

† English translations of these papers are available from the Real Analysis Exchange at no cost.

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