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Monotonicity Theorems for Approximate Peano Derivatives and Integrals

1. Introduction and notations.

In connection with Burkill's Cesaro-Perron integrals, certain sufficient conditions for functions to be monotone are listed and discussed, and some related problems are also stated.

The main notions to be involved are those of approximate and ordinary Peano derivatives and those of generalized absolutely continuous functions. Some references related to the notions will be given later in the context. Here we first fix the notations.

Let F be a (real-valued) function of a real variable. For a positive integer n, the $n^{\underline{th}}$ approximate (ordinary, resp.) of F at a point x, if it exists, will be denoted as $F_{(n)}(x)$ ($F_n(x)$, resp.). The lower $n^{\underline{th}}$ approximate Peano derivate of F at x, if it exists, will be denoted as $\ell F_{(n)}(x)$, and the upper one as $uF_{(n)}(x)$. The corresponding ordinary ones will be denoted as $\ell F_n(x)$, $uF_n(x)$. If in the definition of $uF_{(n)}(x)$, the approximate limsup is replaced by the ordinary limsup, then an extended real number not less than $uF_{(n)}(x)$ is defined which will be denoted as $u_0F_{(n)}(x)$ (see [7]).

A function F is said to be lower absolutely continuous (or simply ℓAC) on a set S if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Sigma[F(b_i) - F(a_i) > -\varepsilon$$

for each set $\{[a_i, b_i]\}$ of finitely many non-overlapping intervals $[a_i, b_i]$ with the end points a_i, b_i in S and such that $\Sigma(b_i - a_i) < \delta$. A function is said to be generalized lower absolutely continuous (or simply [ℓACG]) on a set if this set is a union of countably many closed sets on each of which the function is ℓAC. A function F is AC ([ACG], resp.) on a set if both F and $-F$ are ℓAC ([ℓACG], resp.) on the set. See [10] for a more detailed discussion for the concept of ℓAC and [ℓACG] etc.

2. Monotonicity theorems.

First we list a general monotone theorem concerning approximate Peano derivatives.

Theorem (1,n). Let $F_{(n)}(x)$ exist finitely for all x in a (bounded) closed interval [a,b]. If $u_0F_{(n+1)}(x) \geq 0$ for almost all x and $u_0F_{(n+1)}(x) > -\infty$ for nearly all x in [a,b], then $F_{(n)}$ is monotone increasing and continuous on [a,b].

Here, by "nearly all x" we mean "for all x except perhaps for those x in a countable set". The theorem, generalizing a result by Verblunsky in [15] for the ordinary Peano derivatives, has been proved in [7]. The proof has been based on Verblunsky's theorem 1, (i), and on a result due to Evans [5], which states that an approximate Peano derivative on a closed interval is a Baire class one function. Theorem (1,n) is more general than what is needed in developing certain integral theories. For convenience of later discussions, we single out the following special case.

Theorem (2,n,A). Let $F_{(n)}(x)$ exist finitely for all x in [a,b]. If $\ell F_{(n+1)}(x) \geq 0$ for almost all x and $\ell F_{(n+1)}(x) > -\infty$ for nearly all x in [a,b], then $F_{(n)}$ is monotone increasing on [a,b].

The even weaker result, obtained by replacing in Theorem (2,n,A) all the <u>approximate</u> Peano derivatives and derivates by the <u>ordinary</u> ones, will be denoted as Theorem (2,n,$\mathcal{O}$).

Now, for convenience, the existence of a finite $F_{(n)}(x)$ ($F_n(x)$, resp.) for n = 0 will simply mean that the function F itself, also denoted as $F_{(0)}$ or F_0, is approximately (ordinarily, resp.) continuous at x. With this convention, it is clear that theorem (1,n) does <u>not</u> hold for n = 0. However, theorem (2,n,A) as well as (2,n,$\mathcal{O}$) remains true for n = 0. In fact, theorem (2,0,A) follows as a corollary of the following interesting result due to Lee in [10].

Theorem 3. Let F be a function defined on [a,b]. Then F is monotone increasing on [a,b] if and only if

(i) F is [ℓACG] on [a,b];

(ii) F is monotone increasing on the closed interval [c,d] whenever it is so on the open interval $(c,d) \subset [a,b]$;

(iii) the upper derivate of F is non-negative almost everywhere in [a,b].

To indicate how Theorem (2,0,A) follows from Theorem 3, we state the following result, which will receive futher discussion in the next section. A proof of this result is straightforward and easy, noting an argument given on page 239-240 in Saks [12] (cf. [9]).

Theorem (4,0,A). Let F be approximately continuous on [a,b]. If $\ell F_{(1)}(x) > -\infty$ for nearly all x in [a,b], then F is [ℓACG] on [a,b].

3. Integrals and problems.

Extending the Perron integral, Burkill [4] has defined a scale of Cesàro-Perron integrals which have been denoted as C_nP-integral for $n = 1,2,3,\cdots$. The original definition for the C_nP-integral has been given in terms of the C_n-continuity and the C_n-derivate which are concepts defined by means of the $C_{n-1}P$-integral, the C_0P-integral being the Perron integral. However, using the Peano derivatives and derivates, the C_nP-integral can be equivalently defined without first defining the $C_{n-1}P$-integral. The fact seems to be implicitly included in Burkill's original work and has been explicitly used by Verblunsky on page 320 in [15]. A complete proof for the fact can be found in Bergin's thesis [2] (cf. also [8]). For convenience, we give the definition of major functions for the C_nP-integral below, using the Peano derivative and derivate.

Let f be a function defined on [a,b]. A function M is said to be a C_n-major function for f on [a,b] if there exists a function F such that

(1) $F_n(x)$ exists finitely and is equal to M(x) for all x in [a,b];

(2) $M(a) = 0$;

(3) $\ell F_{n+1}(x) > -\infty$ for all x in [a,b];

(4) $\ell F_{n+1}(x) \geq f(x)$ for all x in [a,b].

Using the C_n-major functions defined above the theory of the C_nP-integral is fundamentally based on the following monotonicity theorem.

Theorem (2',n,$\mathcal{P}$). Let $F_n(x)$ exist finitely for all x in [a,b] and $\ell F_{n+1}(x) \geq 0$ for all x in [a,b]. Then F_n is monotone increasing on [a,b].

As having pointed out in [2] and [14], Burkill's original proof for this fundamental theorem is defective, but the theorem is correct. In fact, stronger results hold as we have listed in the previous section (cf. Theorem (2,n,A).

and $(2,n,\mathcal{O})$.

If in the definition of the C_n-major function the inequality in (3) is only required to hold for "nearly all x" and that in (4) for "almost all x", then a <u>modified</u> C_n-major function is defined. Using modified C_n-major functions and Theorem $(2,n,\mathcal{O})$, a <u>modified</u> C_nP-integral is obtained. It is a known and interesting result (see page 162 in [3]) that the C_nP-integral and the modified one are equivalent.

Now, we come to approximate extensions of the C_nP-integral. In the definition of the C_n-major function, replacing the ordinary Peano derivatives and derivates by the corresponding approximate ones, an approximate C_n-major function is defined. Then using Theorem (2,n,A) an approximate C_nP-integral is defined, which will be denoted as A_nP-integral. Similarly, a modified A_nP-integral is defined. It is clear that the modified A_nP-integral is more general than the A_nP-integral. But whether or not these two integrals are equivalent is an open question. We list it as a problem below. We remark that the modified A_nP-integral has been discussed in better detail in [7].

Problem 1. Are there functions which are modified A_nP-integrable but not A_nP-integrable?

To state another problem, let us note that the C_nP-

integral is equivalent to the C_nD-integral which was beautifully developed by Sargent in [13]. The main concept involved in the C_nD-integral is that of a generalized absolutely continuous (in C_n-sense) function. Such a function as defined by Sargent [13] will be simply termed as an AC_nG^* function. It may be helpful to note that an AC_nG^* function may also be equivalently defined by using the Peano derivatives instead of the C_{n-1}P-integral. It is easier to think that an AC_nG^* function is just a natural $n^{\underline{th}}$ Peano analogue of an ACG* function as given in Saks [12], which is essential for a descriptive definition of the Denjoy integral in the restricted sense. The equivalence of the C_nP- and C_nD-integral is just a natural extension of that of the Perron integral and the Denjoy integral in the restricted sense. We write the equivalence relation in the "if and only if" form, noting that a complete proof for the "only if" part has been just produced recently by Verblunsky in [14].

Theorem $(5,n,\theta)$. A function f defined on [a,b] is C_nP-integrable over [a,b] if and only if there exists a function F such that the $n^{\underline{th}}$ Peano derivative, F_n, of F is AC_nG^* on [a,b] and such that $F_{n+1}(x)$ exists finitely and is equal to f(x) for almost all x in [a,b]. In this case, F_n is an indefinite C_nP-integral of f on [a,b].

Now it is natural to ask whether or not a similar result for the A_nP-integral or even for the modified A_nP-integral can be obtained. We list it as a problem below.

Problem (5,n,A). Can a suitable notion similar to that of an $AC_n^-G^*$ function be defined so that a result for the A_nP-integral similar to Theorem (5,n,$\mathcal{O}$) holds?

To attack the problem, further investigation about the C_nP- and C_nD-integral might be useful. Note first that an AC_nG^* function is [ACG]. Therefore, it follows from Theorem (5,n,$\mathcal{O}$) that an indefinite C_nP-integral is [ACG]. Then, using Theorem (2,n,$\mathcal{O}$) and the definition of C_n-major functions, one proves easily that a C_n-major function of a C_nP-integrable function is [ℓACG]. However, whether or not the following problem has an affirmative answer is an open question for $n \geq 2$.

Problem (4,n,$\mathcal{O}$). Is it true that if $F_n(x)$ exists finitely for all x in [a,b] and if $\ell F_{n+1}(x) > -\infty$ for nearly all x in [a,b], then F_n is [ℓACG] on [a,b]?

Of course, replacing the ordinary Peano derivative and derivate by the approximate ones, one has a more general problem, which is being denoted as Problem (4,n,A).

For $n = 0$, the answer to problem (4,n,A) (and hence also the problem (4,n,$\mathcal{O}$)) is in the affirmative due to Theorem (4,0,A). A closer study of the interesting paper by Jeffery [6] also shows that problem (4,1,0) has an affirmative answer. This we list as Theorem (4,1,0) below.

Theorem (4,1,$\mathcal{O}$). If F is a function on [a,b] such that F_1 exists finitely on [a,b] and such that $\ell F_2(x) > -\infty$ for nearly all x in [a,b], then F_1 is [ℓACG] on [a,b].

With Theorem (4,1,$\mathcal{O}$) at hand, it is noted that Theorem (2,1,$\mathcal{O}$), similar to Theorem (2,0,A), is just a special case of Theorem 3. The same would be true for Theorem (2,n,$\mathcal{O}$) for $n \geq 2$ ((2,n,A) for $n \geq 1$) should the answer to problem (4,n,$\mathcal{O}$)((r,n,A)) be in the affirmative.

Bypassing the problems (4,n,A) and (5,n,A), people have directly used [ℓACG] functions to define more general integrals. Many integrals so defined are extensions of the Denjoy integral in the wide sense. It is interesting to note that for these integrals, results similar to the theorem of Hake-Alexandroff-Looman have been established. In fact, based on Theorem 3 listed in the previous section, an abstract integral of Perron type and one of Denjoy type have been defined and proved to be equivalent in [10]. Some concrete integrals, old and new, have been also given there as examples of the abstract theory. It is noted that the equivalent relation proved in [10] could be used to supply another proof of the "only if" part of Theorem (5,n,$\mathcal{O}$) should the answer to problem (4,n,$\mathcal{G}$) be in the affirmative.

We conclude this note by mentioning that Evans [5], Babcock [1], O'Malley and Weil [11] and many others have obtained many interesting properties for the approximate or ordinary Peano derivatives. Their results might be useful in attacking the problems stated here.

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