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# Nowhere Monotone Functions

Let $M$ denote the class of functions of a real variable which are monotone on a measurable set of positive measure. Burkill and Mirsky asked in [1,p.408] whether there is a differentiable function which is not in $M$. Motivation for this question comes from [2] where it is shown that every continuous function is monotone on some perfect set, from [3,p.412] where a differentiable function which is not monotone on any interval is constructed, and from [5] where, for every $\varepsilon > 0$, it is shown that there is an infinitely differentiable function which is not monotone on any set with measure greater than $\varepsilon$. That every differentiable function belongs to $M$ is a consequence of the theorems below. An example of a continuous function not in $M$ is given in [5] by constructing a continuous function which is nowhere approximately derivable and proving that any function which is almost everywhere not approximately derivable is not in $M$. That there exist functions of bounded variation which are not in $M$ is shown by the example below. However, it follows from Theorem 2 that any bounded variation function which is not in $M$ must be singular (i.e., its derivative is equal to 0 almost everywhere).

Theorem 1. If $f(x)$ is continuous and satisfies Lusin's condition (N) on an interval $I$ (i.e., the image of every set of measure 0 is of measure 0), then $f(x)$ is monotone on a perfect set of positive measure.

Theorem 2. If $f(x)$ is a function of a real variable, let $E_f = \{x | f'(x)$ exists and $0 < |f'(x)| < \infty\}$. Then $E_f$ is measurable and, if $E_f$ is of positive measure, then $f(x)$ is monotone on a subset of $E_f$ of positive measure.

Remark. If $f$ is not a constant function, but is differentiable, then $f$ satisfies the hypotheses of both of these theorems. Hence, differentiable functions are always monotone on sets of positive measure.

Proof of Theorem 1. If $f$ is not identically constant, there exist $a,b \in I$ such that $f$ takes on its minimum at $a$ and its maximum at $b$. Without loss of generality, $a < b$. For each $y \in f([a,b])$, let $x(y) = \inf\{x \in [a,b]: y = f(x)\}$. Let $A = \{x | x = x(y)\}$. Following [4,p.283], $A = [a,b] \setminus \bigcup_n E_n$ where

$$E_n = \{x \in [a + \frac{1}{n},b]: \exists t \in [a,b] \text{ with } f(t)=f(x) \text{ and } x-t \geq \frac{1}{n}\}.$$

From the continuity of $f$, it follows that each $E_n$ is closed and thus $A$ is measurable. Again, from the continuity of $f$, it follows that $f$ is monotone increasing on $A$. For if $x_1,x_2 \in A$, $x_1 < x_2$, and $f(x_1) \geq f(x_2)$, then by the intermediate value property it follows that $\exists x_3 \in [a,x_1]$ such that $f(x_3) = f(x_2)$. But this contradicts the fact that

$$x_2 = \inf\{x \in [a,b]: f(x) = f(x_2)\}.$$

Since $f(A) = f([a,b])$ is of positive measure and $f$ satisfies Lusin's condition (N), it follows that $A$ is of positive measure. Thus the theorem is proved.

Proof of Theorem 2. That $E_f$ is measurable follows from the fact that for every function $f$ of a real variable,

$$\overline{D}f(x) = \overline{\lim_{h\to 0}} \frac{f(x+h)-f(x)}{h} \text{ and } \underline{D}f(x) = \underline{\lim_{h\to 0}} \frac{f(x+h)-f(x)}{h}$$

are measurable functions. (cf. [4,p.112]. Then

$$E_f = \{x|\overline{D}f(x) = \underline{D}f(x) \text{ and } \overline{D}f(x) \in (-\infty,0)\cup(0,\infty)\}$$

is Lebesque measurable. Without loss of generality, suppose $E^+ = \{x|0 < f'(x) < \infty\}$ is of positive measure. For each natural number $n$ and integer $k$, let $E_{nk}$ be the set of $x \in E^+ \cap [\frac{k}{n}, \frac{k+1}{n}]$ which satisfy $\frac{f(x)-f(y)}{x-y} > 1/n$ whenever $|x-y| \le 1/n$. Then, for $x_1,x_2 \in E_{kn}$, $x_1 < x_2$ implies $|x_1-x_2| \le 1/n$ which in turn implies that $f(x_2) - f(x_1) > (x_2-x_1)/n$. Thus $f$ is monotone increasing on each $E_{kn}$. Since $E^+ = \bigcup_k \bigcup_n E_{kn}$ it follows that one of the $E_{kn}$ is of positive measure. Thus $f$ is monotone on a set of positive measure and the theorem is proved.

Example. A continuous function of bounded variation defined on $[0,1]$ which is not monotone on any measurable set of positive measure.

Construction. Let $C$ be the Cantor ternary set; i.e.,

$$C = \{x \in [0,1]: x = \Sigma\, x_i/3^i \text{ where each } x_i = 0 \text{ or } 2\}.$$

Let $c(x)$ be the Cantor singular function; i.e., if $x \in C$ and $x = \Sigma\, x_i/3^i$, then $c(x) = \Sigma\, x_i/2^i$ and $c(x)$ is linear on intervals contiguous to $C$. Let

$$y_i(x) = \begin{cases} 0 & \text{if } x = \Sigma\, x_j/3^j \text{ where } x_i = 0 \text{ or } x_i = 2 \\ 1 & \text{if } x = \Sigma\, x_j/3^j \text{ where } x_i = 1 \end{cases}$$

Let $h(x) = \min(c(x), 1-c(x))$ and extend $h(x)$ to the entire real line so that $h(x) = h(x+1)$. Let $f_i(x) = 8^{-i} y_i(x) \cdot h(3^i x)$. It is readily observed that each $f_i(x)$ is continuous. Furthermore, since $y_i(x) = 0$ except on $3^i$ intervals, it follows that $f_i(x)$ has variation in $[0,1]$ of magnitude $2 \cdot 8^{-i} \cdot 3^i$. Thus $\mathrm{Var}(f_i) = 2 \cdot (3/8)^i$. Let $f(x) = \Sigma\, f_i(x)$. Then $\mathrm{Var}(f) \leq \Sigma\, \mathrm{Var}(f_i) = 2 \cdot 3/5 < \infty$. Let $P$ be a set of positive measure, $E$ be the set of real numbers which have infinitely many 1's in their ternary expansion, and $P' = P \cap E$. Then, since $|E^c| = 0$, $|P'| = |P|$. Let $x$ be a point of density of $P'$ and let

$$I = [a,b] = [.x_1 x_2 \ldots x_n 0,\ .x_1 x_2 \ldots x_n 1]$$

satisfy $x \in I$ and $|P' \cap I| > 8/9|I|$. Then, by the construction of $f_i$, for $i = 1,2,\ldots n$ each $f_i$ is constant on $I$.

Since $|P' \cap I| > 8/9|I|$ there are points $a_1, a_2, a_3 \in P'$ such that

$$a_1 \in (a + 1/9\, d,\ a + 2/9\, d)$$

$$a_2 \in (a + 1/3\, d,\ a + 2/3\, d)$$

$$a_3 \in (a + 7/9\, d,\ a + 8/9\, d)$$

where $d = b-a$. Then $f_{n+1}(a_1) = f_{n+1}(a_3)$ and $f_{n+1}(a_2) - f_{n+1}(a_1) = \frac{1}{2}\, 8^{-n-1}$. Hence

$$|f(a_3) - f(a_1)| \leq 2 \cdot \sum_{n+2}^{\infty} 8^{-i} = 2/7 \cdot 8^{-n-1}$$

and

$$|f(a_2) - f(a_1)| > \frac{1}{2}\, 8^{-n-1} - 2 \sum_{n+2}^{\infty} 8^{-i} = 5/14 \cdot 8^{-n-1}.$$

It follows that $f$ is not monotone on $P$ and since $P$ was an arbitrary perfect set of positive measure, it follows that $f$ is not monotone on any measurable set of positive measure.

## References


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