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ON THE SPACE $BSV^m[a,b]$

The statement that f has bounded slope variation with respect to m over [a,b] means that f is a function whose domain includes [a,b], m is a real-valued increasing function on [a,b], and there exists a non-negative number B such that if $\{x_i\}_{i=0}^{n}$ is a subdivision of [a,b] with n > 1, then

$$(1)\quad \sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)} - \frac{f(x_i) - f(x_{i-1})}{m(x_i) - m(x_{i-1})} \right| \leq B.$$

The least such number B is called the slope variation of f with respect to m over [a,b] and is denoted by $V_a^b(df/dm)$. [Note: $V_a^a(df/dm) = 0$.] $BSV^m[a,b]$ is the space to which f belongs if and only if f has bounded slope variation with respect to m over [a,b].

In case m is continuous on [a,b], then $BSV^m[a,b]$ is the space to which F belongs if and only if there is a function f in BV[a,b] such that, for each x in [a,b], $F(x) = \int_a^x f\,dm + F(a)$, where the integral is the Stieltjes integral ([2], Theorem 3). No such characterization has been obtained for the case when m is not continuous on [a,b].

Theorem 2 of [3] shows that $BSV^m[a,b]$ is a subset of BV[a,b]. However, $BSV^m[a,b]$ with the BV-norm $\|f\|_v = V_a^b(f) + |f(a)|$ is not complete.

THEOREM 1. $BSV^m[a,b]$ <u>with norm</u> $\|f\|_{sv}^m = V_a^b(df/dm) + |D_m^- f(b)| + |f(a)|$ <u>is a Banach space, where</u> $D_m^- f(b)$ <u>denotes the left-hand derivative of</u> f <u>with respect to</u> m <u>at</u> b.

If m is an increasing function on [a,b], an m-polygonal function on [a,b] is a generalization of a polygonal function obtained by replacing straight line segments with arcs of the curve $y = m(x)$. Every m-polygonal function on [a,b] belongs to $BSV^m[a,b]$.

THEOREM 2. <u>If</u> f <u>is in</u> $BSV^m[a,b]$, <u>there exists an infinite sequence</u> $\{\theta_n\}$ <u>of m-polygonal functions on</u> [a,b] <u>which converges to</u> f <u>on</u> [a,b] (<u>If</u> f <u>is continuous on</u> [a,b], <u>the convergence is uniform.</u>) <u>and such that as</u> $n \to \infty$, <u>the infinite sequences</u> $\{V_a^b(d\theta_n/dm)\}$, $\{V_a^b(\theta_n)\}$, $\{\int_a^b (d\theta_n)^2/dm\}$ <u>and</u> $\{\int_a^b \theta_n\, dm\}$ <u>converge respectively to</u> $V_a^b(df/dm)$, $V_a^b(f)$, $\int_a^b (df)^2/dm$ <u>and</u> $\int_a^b f\, dm$, <u>where</u> $\int_a^b (df)^2/dm$ <u>is the Hellinger integral and</u> $\int_a^b f\, dm$ <u>is the mean Stieltjes integral.</u>

It is a consequence of Theorem 2 of [3] that a Cauchy sequence in $BSV^m[a,b]$ with norm $\|\cdot\|_{sv}^m$ is also a Cauchy sequence in BV[a,b] with norm $\|\cdot\|_v$. In fact, a slight modification of the proof of Theorem 1 gives the following result.

THEOREM 3. $BSV^m[a,b]$ *with norm* $\|f\|_{svm} = V_a^b(df/dm) + V_a^b(f) + |f(a)| = V_a^b(df/dm) + \|f\|_v$ *is a Banach space.*

Roberts and Varberg [5] and G. F. Webb [7] have considered the special case $m(x) = x$ and have shown $BSV^m[a,b]$ is a Banach space with a norm which, in each case, differs from both $\|\cdot\|_{sv}^m$ and $\|\cdot\|_{svm}$.

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