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Symmetric Functions

We say that a real function f is *symmetric at* x provided there exists ϵ_x such that $h < \epsilon_x \rightarrow f(x-h) = f(x+h)$. If f is symmetric at all x, then we say f *is symmetric*.

Theorem 1. If f is symmetric, and f is constant on some interval, then f is constant except on a nowhere dense countable set.

Proof. Suppose $f(x) = A$ on (a,b) and suppose $\{x: f(x) \neq A\}$ is dense on some interval (c,d). Without loss of generality, let $c \geq b$. Define $x_0 = \inf \{x: (x,d) \cap \{x: f(x) \neq A\}$ is dense$\}$. Then $(x_0,d) \cap \{x: f(x) \neq A\}$ is dense. But there exists ϵ_{x_0} such that $h < \epsilon_{x_0} \rightarrow f(x_0 - h) = f(x_0 + h)$. Thus, the following set is dense: $(x_0 - \epsilon_{x_0}, d) \cap \{x: f(x) \neq A\}$ This contradicts the definition of x_0, so $\{x: f(x) \neq A\}$ is nowhere dense.

Now suppose $U = \{x: f(x) \neq A\}$ is uncountable. Without loss of generality, assume $U \cap (b,\infty)$ is uncountable. Let x_0 be the infimum of the set of condensation points of $U \cap (b,\infty)$. Then x_0 is a condensation point of $U \cap (b,\infty)$, and x_0 is a limit point of condensation points. Also, there exists ϵ_{x_0} such that $h < \epsilon_{x_0} \rightarrow f(x_0 - h) = f(x_0 + h)$ Thus there is a condensation point of $U \cap (b,\infty)$ in $(x_0 - \epsilon_{x_0}, x_0)$. This contradicts the definition of x_0. Thus $\{x: f(x) \neq A\}$ is countable.

Theorem 2. *If f is symmetric, but is not constant on some interval, then f must take on at least two values, A and B, such that both $f^{-1}(A)$ and $f^{-1}(B)$ contain c points in every interval.*

Proof. Define $E_n = \{x : \epsilon_x > 1/n\}$. Since the countable union of E_n's is the line, some E_n has c points in an interval (a,b) of length $< 1/n$. Let $f(a) = A$. Then $f^{-1}(A)$ has c points in $(a, b+(b-a))$ by symmetry of the c points of E_n in (a,b). That is, for every x in $E_n \cap (a,b)$, $x+(x-a)$ is in $(a, b+(b-a))$, and $f(x+(x-a)) = f(x-(x-a)) = f(a) = A$. Suppose that $f^{-1}(A)$ has $< c$ points in some interval (d,e). Without loss of generality, assume $f^{-1}(A)$ has c points in $(-\infty, d)$. Let x_0 be the infimum of the set of x for which $(x,e) \cap f^{-1}(A)$ has $< c$ points. Then (x_0,e) has $< c$ points since $(x_0,e) \cap f^{-1}(A) = \bigcup_n (x_0 + 1/n, e) \cap f^{-1}(A)$ is a countable union of sets each having $< c$ points, so has $< c$ points. Now, there exists ϵ_{x_0} such that $f(x_0 - h) = f(x_0 + h)$ for all $h < \epsilon_{x_0}$. Thus, $(x_0 - \epsilon_{x_0}, e) \cap f^{-1}(A)$ has $< c$ points, contradicting the definition of x_0. Thus, $f^{-1}(A)$ has c points in every interval.

If f is not constant on some interval, f must take on another value, B, in (a,b). Then $f^{-1}(B)$ has c points in $(a-(b-a), b+(b-a))$ and the same arguments apply as above, so that $f^{-1}(B)$ has c points in every interval.

Theorem 3. *If f is symmetric and f is not constant on some interval, then f must take on two values, A and B, such that for every $\epsilon > 0$, $f^{-1}(A)$ and $f^{-1}(B)$ each have outer measure $\geq (b-a)(1-\epsilon)$ in some interval, (a,b).*

Proof. Given $\epsilon > 0$, define $E_n = \{x: \epsilon_x > 1/n\}$. Some E_n has positive outer measure. Thus there exists (a,b) such that $(b-a) < 1/n$ and $m^*(E_n \cap (a,b)) > (b-a)(1-\frac{\epsilon}{2})$. Let $z = (a+b)/2$. Then $m^*(E_n \cap (a,z)) \geq \frac{1}{2}(b-a)(1-\frac{\epsilon}{2})$ or $m^*(E_n \cap (z,b)) \geq \frac{1}{2}(b-a)(1-\frac{\epsilon}{2})$. Assume the first without loss of generality. Define f(a) = A. Then $m^*(f^{-1}(A) \cap (a,b)) \geq (b-a)(1-\frac{\epsilon}{2})$ because x in $E_n \cap (a,z)$ implies $x + (x-a)$ or $2x-a$ is in $f^{-1}(A) \cap (a,b)$. Now choose u within $\frac{\epsilon}{4}(b-a)$ of a so that $f(u) = B \neq A$. Then

$$m^*(f^{-1}(B) \cap (a,b)) \geq (b-a)(1-\frac{\epsilon}{2}) - 2\frac{\epsilon}{4}(b-a).$$

So $m^*(f^{-1}(B) \cap (a,b)) \geq (b-a)(1-\epsilon)$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ have outer measure $\geq (b-a)(1-\epsilon)$ in the interval (a,b).

Theorem 4. If f is measurable, and symmetric, then f is constant except on a nowhere dense countable set.

Proof. From Theorem 3, if f is measurable, $f^{-1}(A)$ and $f^{-1}(B)$ each have measure $\geq 3/4(b-a)$ on (a,b), which implies A = B. Thus f is constant in some interval, and Theorem 1 implies the result.

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