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Products of Approximate Derivatives

In (2) and (3) a characterization is given for the class of functions defined on $[0,1]$ whose product with every derivative is a derivative. In the present note characterizations of the multiplier class for approximate derivatives of continuous functions (ADC) and the multiplier class for the approximate derivatives (AD) are given and their proofs are sketched.

For a function $F(x)$, let $W(F,I)$ denote its total variation on the interval I and let $W(t) = W(F, [0,t])$.

Theorem 1. A function $F(x)$ belongs to ADC if and only if $W(t)$ satisfies a Lipschitz condition at each point x in $[0,1]$.

(Stated in the form of a Stieltjes integral, this condition requires that $\left|\int_{x_0}^{x} dW(t)\right| \leq M_{x_0} \cdot |x - x_0|$.)

It is interesting to compare this with the notion of distant bounded variation, $\left|\int_{x_0}^{x} (t - x_0) dW(t)\right| \leq M_{x_0} \cdot |x - x_0|$, which characterizes the multiplier class for ordinary derivatives.

Proof. Sufficiency is established by taking the approximate derivative of both sides of the integration by parts formula for the wide sense Denjoy Integral, (4, p. 246). After replacement of "limit" by "approximate limit", the argument is identical to the proof that the product of a continuous function of bounded variation with an ordinary derivative is an ordinary derivative, (2, p. 174).

To show necessity we assume that $W(t)$ fails to satisfy a Lipschitz condition at $x = 0$ and construct an approximate derivative $g(x)$ (with continuous primitive) such that $F(x)g(x)$ is not the approximate derivative of its wide sense Denjoy integral. This construction uses the same techniques used in the construction of the counterexample for ordinary derivatives in (3) and the following lemma.

Lemma. If $W(t)$ fails to satisfy a Lipschitz condition at $x = 0$, then there exists a sequence x_n tending to 0 and a sequence of intervals L_n tending to 0 such that 0 is a point of dispersion of $E = \bigcup L_n$ and $W(F, [x_{n+1}, x_n] \cap E) > n \cdot x_n$.

<u>Proof</u>. We first note that if $J_n = [2^{-n}, 2^{-n+1}]$, $\{e_n\}$ is a non-increasing sequence whose limit is 0 , and I_n is any subinterval of J_n such that $|I_n| = e_n \cdot |J_n|$, then 0 is a point of dispersion of $E = \bigcup I_n$ since if x belongs to J_N,

$$|[0,x] \cap E|/x \leq e_N 2^{-N+1}/x \leq 2^N e_N 2^{-N+1} = 2e_N .$$

Since $W(t)$ does not satisfy a Lipschitz condition at $x = 0$, we may choose x_1 such that $W(x_1) > 2x_1$. For some k , $x_1 \in J_k$. Divide the intervals $[2^{-k}, x_1]$, J_{k+1}, J_{k+2}, ... in half and begin choosing the sequence $\{L_n\}$ by taking the half of each of these intervals on which the variation of F is greater than or equal to ½ the variation of F on the contaning interval. Then for some finite number m_1 ,

$$W(F, \bigcup_{i=1}^{m_1} L_i^1) > ½ \cdot 2x_1 = x_1 .$$

After $x_1, \ldots, x_{n-1}$ and their corresponding intervals L have been selected, choose $x_n < a/2$, where a is the infimum of the previously chosen L and such that $W(x_n) > n^2 x_n$. For some k, $x_n \in J_k$ and we subdivide the intervals $[2^{-k}, x_n]$, J_{k+1} , J_{k+2} , ... into n equal parts and choose the intervals L to be an n-th of the containing interval J on which $W(F,L) \geqslant (1/n)\cdot W(F,J)$. Then after a sufficiently large (but finite) number of the L , L_1^n , ... , $L_{m_n}^n$, have been chosen, $W(F, \bigcup_{i=1}^{m_n} L_i^n) \geqslant (1/n)n^2 x_n = n x_n$, and the proof of the lemma is complete.

Then if $L \in \{L_i\}$ and $L \subset [x_{n+1}, x_n]$, by (3, Lemma 1) we define $g(x)$ to be a piecewise linear, continuous function on $L = [a,b]$ such that

$$\text{(i)} \quad g(a) = g(b) = 0$$

$$\text{(ii)} \quad \int_a^b g(x)dx = 0$$

$$\text{(iii)} \quad W\left(\int_a^x g(t)dt , L \right) < 1/n$$

$$\text{(iv)} \quad \int_a^b F(x)g(x)dx > (1/n)\cdot W(F,L) .$$

Set $g(x) = 0$ for $x \notin E = \bigcup L_i$. Condition (i) insures that $g(x)$ is continuous on $(0,1]$. Conditions (ii) and (iii) yield the wide sense Denjoy integrability of $g(x)$ on $[0,1]$ and also, letting $G(x) = (D)\int_0^x g(t)dt$, then $G(x) = 0$ if $x \notin E$. Since 0 is a point of dispersion of E, $G'_{ap}(0) = g(0) = 0$. Combining (iv) and the Lemma, we have that

$$\int_0^{x_n} F(x)g(x)dx \geqslant \int_{x_{n+1}}^{x_n} F(x)g(x)dx \geqslant (1/n)\, n\, x_n = x_n .$$

Noting that $g(x) = 0$ on $[x_n, 2x_n]$, $(x_n < a/2)$, we have that for $x \in [x_n, 2x_n]$,

$$(1/x)\int_0^x F(t)g(t)dt = (1/x)\int_0^{x_n} F(t)g(t)dt \geqslant$$

$$(1/x)\int_{x_{n+1}}^{x_n} F(t)g(t)dt \geqslant (1/2x_n)\int_{x_{n+1}}^{x_n} F(t)g(t)dt \geqslant (1/2) .$$

Consequently, the set of points x at which

$$(1/x)\int_0^x F(t)g(t)dt \geqslant 1/2$$

has an upper density of at least ½ at $x = 0$. Thus, $(\int F(t)g(t)dt\,)'_{ap} \neq F(0)g(0) = 0$ and $F(x)g(x)$ is not the approximate derivative of a continuous function. This completes the proof.

<u>Theorem 2</u>. <u>A function</u> $F(x)$ <u>belongs to</u> AD <u>if and only if</u> $F(x)$ <u>is constant on</u> $[0,1]$.

Sufficiency is obvious. Necessity can be established by a very tedious process of tracking down and labeling a sequence of intervals on which $F(x)$ varies and building $g(x)$ as above (the primitive is only approximately continuous in this case).

However, the following example suggests that there should be a resonably simple proof of Theorem 2 which has so far eluded this author.

<u>Example</u>. <u>The identity function does not belong to</u> AD.

<u>Proof</u>. Let $I_n = [a_n, b_n]$ be a sequence of intervals tending to 0 such that 0 is a point of dispersion of $E = \bigcup I_n$. On I_n let $G(x)$ be a positive, differentiable function such that $G(a_n) = G(b_n) = G'(a_n) = G'(b_n) = 0$ and such that $\int_{a_n}^{b_n} G(x)dx = 1$. Set $G(x) = 0$ for $x \notin E$.

By a result proved in (1), if $G(x)$ were an approximate derivative, it would be an ordinary derivative because $G(x) \geqslant 0$. Since $\int_0^1 G(x)dx = +\infty$, it's neither.

Since $G(x) = 0$ for $x \notin E$, both $G(x)$ and $xG(x)$ are approximately derivable at $x = 0$ and their approximate derivatives have the value 0 there. In particular

$$g(x) = \begin{cases} G'(x) & ; x > 0 \\ 0, & ; x = 0 \end{cases} \quad \text{and} \quad (xG(x))'_{ap} \quad \text{are}$$

approximate derivatives on $[0,1]$ and the relation

$$(xG(x))'_{ap} = G(x) + xg(x)$$

holds on $(0,1]$ by the product rule and at $x = 0$ since all three terms are equal to 0. Since $(xG(x))'_{ap}$ is an approximate derivative and $G(x)$ is not, $xg(x)$ is not an approximate derivative and $F(x) = x$ does not belong to AD.

Bibliography

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