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Directional Cluster Sets

Let f be a function from the open upper half plane H into the Riemann sphere W, and let x be a point on the real line R. The *cluster set* C(f,x) *of* f *at* x and the *essential cluster set* $C_e(f,x)$ *of* f *at* x are defined as follows: the point $w \in W$ is in C(f,x) [resp., $C_e(f,x)$] if x is a point of accumulation [resp. a point of positive upper density] of $f^{-1}(U)$ for every open neighborhood U of w. The *cluster set* $C(f,x,\theta)$ and the *essential cluster set* $C_e(f,x,\theta)$ *of* f *at* x *in the direction* θ $(0<\theta<\pi)$ are defined analogously in the obvious manner. We set

$$\Theta(x) = \{\theta: C(f,x) = C(f,x,\theta)\}$$

and

$$\Theta^*(x) = \{\theta: C_e(f,x) \subset C_e(f,x,\theta)\}.$$

E. F. Collingwood established the following result [6, Theorems 2 and 3].

Theorem C. *If* $f: H\to W$ *is continuous, then* $\Theta(x)$ *is residual at each point* x *of a residual subset* S *of* R.

Concerning this result, A. M. Bruckner and Casper Goffman [5] have raised the question: Can the residual set $\Theta(x)$ be taken to be the same for every $x \in S$? We have answered this question in the negative by establishing the following result [4].

Theorem 1. *There exists a continuous* f: H→W *such that* $\bigcap_{x \in S} \theta(x)$ *is a first category set of directions for each residual subset* S *of* R.

The construction of this function used certain sets of J.-P. Kahane [8] as building blocks.

Casper Goffman and W. T. Sledd established the following result [7, Theorem 2].

Theorem GS. *If* f: H→W *is measurable and* θ *is a direction, then*

$$C_e(f,x) \subset C_e(f,x,\theta) \quad ,$$

except for a set of measure zero; furthermore, if f *is continuous, then*

$$C_e(f,x) \subset C_e(f,x,\theta) \quad ,$$

except for a set of the first category.

To supplement this, we have established the following result [3].

Theorem 2. *Let* f: H→W. *If* f *is measurable, then* $|\theta^*(x)| = \pi$ *for almost every and nearly every* $x \in R$; *furthermore, if* f *is continuous, then* $\theta^*(x)$ *is residual for almost every and nearly every* $x \in R$.

Again, a natural question to ask is whether or not, for a given function f, there exists a "large" set of directions Θ^* such that $\Theta^* \subset \Theta^*(x)$ for a "large" set of points $x \in R$. As a partial answer, we have proved the following result [4].

Theorem 3. *There exists a continuous* f: $H \to W$ *such that the intersection* $\bigcap_{x \in S} \Theta^*(x)$ *is* (a) *of the first category if* $S \subset R$ *is residual, and* (b) *of measure zero if* $S \subset R$ *is of full measure.*

By consolidating results of F. Bagemihl, G. Piranian, and G. S. Young [2, Theorem 6] and Bagemihl [1, Theorem 11], we obtain

Theorem BPY. *Let* f: $H \to W$ *be holomorphic. Then to almost every and nearly every* $x \in R$, *there corresponds a set* $A = A(x)$ *of directions whose complement contains at most one direction and for which* $\bigcap_{\theta \in A} C(f,x,\theta) \neq \emptyset$.

This theorem is not true for continuous functions. However, by combining Theorem C with a result of P. Lappan [9, Theorem 1], we arrive at the following analogue of Theorem BPY for continuous functions.

Theorem CL. *Let* f: $H \to W$ *be continuous. Then for almost every and nearly every* $x \in R$, *there corresponds a set* $B = B(x)$ *of directions whose complement is of the first category and for which* $\bigcap_{\theta \in B} C(f,x,\theta) \neq \emptyset$.

We note that this theorem is also a direct consequence of Theorem 2; furthermore, Theorem 2 yields the following result which supplements both of the theorems cited above.

Theorem 3. Let $f\colon H \to W$ be measurable. Then to almost every and nearly every $x \in R$, there corresponds a set $C = C(x)$ of directions whose complement is of measure zero and for which $\bigcap_{\theta \in C} C(f,x,\theta) \neq \emptyset$.

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