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Some Uniform Pathology for Borel Measures

The results pointed out here constitute a natural "soft extension" of the main content of [2]; details will shortly appear in an appendix section of the first author's forthcoming joint monograph [1] with H.R. Bennett. In what follows, T is a second countable topological space of cardinality $2^{\aleph_0}$ (e.g., the real line), μ is a complete, regular, sigma-finite, non-atomic Borel measure in T such that $\mu(T) > 0$, μ^* is the outer measure induced by μ, and μ_* is the customary inner measure. P(T) denotes the set of all subsets of T. For information concerning Martin's Axiom, see [3].

Definition. <u>A function $\mathcal{V}$: $P(T) \to P(T)$ is *well behaved* if</u> (<u>a</u>) <u>$\mathcal{V}(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} \mathcal{V}(X_i)$ (so that $\mathcal{V}$ is determined by its action on singletons)</u>, (<u>b</u>) <u>$\mathcal{V}(\emptyset) = \emptyset$</u>, <u>and</u> (<u>c</u>) <u>$(\forall X)(\forall Y)[(\mathcal{V}(X) \subseteq X \;\&\; X \cap Y = \emptyset) \Rightarrow X \cap \mathcal{V}(Y) = \emptyset]$</u>. <u>$\mathcal{V}$ is *small* if $\mathrm{card}(\mathcal{V}(X)) \leq \aleph_0 \cdot \mathrm{card}(X)$ holds for all $X \subseteq T$</u>. <u>A set $X \subseteq T$ is *$\mathcal{V}$-invariant* if $\mathcal{V}(X) \subseteq X$</u>.

Examples of small well-behaved functions which are of interest in classical measure theory are (1) the $\mathcal{X}$ arising from rational translation on the line and (2) the $\mathcal{X}$ arising from allowing disagreement at finitely many coordinates in the space $\{0,1\}^Z$. Of course, the identity function on P(T) is also small and well behaved.

Theorem 1 (cf. [2]). Suppose T and μ satisfy the additional condition: $(\forall X \subseteq T)[(\mu(X)$ defined and $> 0) \Rightarrow \text{card}(X) = 2^{\aleph_0}]$. Let $\mathcal{X}$ be a small well behaved function from P(T) into P(T); and let A be an $\mathcal{X}$-invariant subset of T such that $\mu(A)$ is defined and is > 0. Then there exists an $\mathcal{X}$-invariant set $B \subseteq A$ such that A-B is $\mathcal{X}$-invariant, $\mu^*(B) = \mu^*(A-B) = \mu(A)$, and $\mu_*(B) = \mu_*(A-B) = 0$.

Theorem 2. Assume no additional condition on T and μ, but assume Martin's Axiom. Let A be an $\mathcal{X}$-invariant subset of T such that $\mu^*(A) > 0$, where $\mathcal{X}$ is small and well behaved. Then there exists an $\mathcal{X}$-invariant subset B of A such that A-B is $\mathcal{X}$-invariant, $\mu^*(B) = \mu^*(A-B) = \mu^*(A)$, and $\mu_*(B) = \mu_*(A-B) = 0$.

References

1. H.R. Bennett and T.G. McLaughlin, A Selective Survey of Axiom-Sensitive Results in General Topology, Texas Tech University Math Series, No. 12, to appear.

2. J. Rosenthal, Nonmeasurable invariant sets, Amer. Math. Monthly, Vol. 82, 1975, 488-491.

3. J.R. Shoenfield, Martin's axiom, Amer. Math. Monthly, Vol. 82, 1975, 610-617.

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