CLUSTER SETS OF ARBITRARY REAL FUNCTIONS: A PARTIAL SURVEY

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Much of the general theory of cluster sets has been developed by Sir Edward Foyle Collingwood, and so it seems appropriate to begin this article with the following paragraph which Collingwood wrote [11, Footnote #2, p.1242] concerning the origin of the theory of cluster sets of arbitrary real functions and its development prior to 1960.

> The theory of the cluster sets of arbitrary real functions originated with W. H. Young. The story begins with his paper [22], in which he showed that the points of inequality of right and left upper and lower limits of a function of a single variable are enumerable. This was followed by a number of papers, some in collaboration with G. C. Young, of which the most important are [24], in which he proved that for a function of a single variable the points of inequality of right and left cluster sets, although not under that or any other compendious name, are enumerable, with analogous theorems for several variables; and [25] which completes and summarizes his theory. Young considered only real functions and was evidently unaware of Painlevé's definition of a cluster set (domaine d'indétermination) which had been formulated in 1895 for complex functions. Perhaps for lack of a suitable terminology and notation to give point to the ideas Young's theorems attracted little notice and, so far as I can discover, have not hitherto been mentioned by writers on complex function theory. I am myself indebted to his daughter, Dr. R. C. H. Tanner, for calling my attention to them. The work of H. Blumberg ([7] and [9]), who had independently discovered Young's theorem of 1908 on discontinuities [5], developed Young's Theory of arbitrary real functions a good deal further and gave rise to the theorems of Jarník [17] and Bagemihl [1] whose well known ambiguous point theorem has important implications for complex function theory.

It is the purpose of this article to attempt to give an explicit and orderly account of the results alluded to in this statement of Collingwood, and then to trace up to the present time subsequent developments related to these results. Even though the original statements of many of these results were made without the terminology of cluster sets, such terminology will be used exclusively here.

Also, the reader is alerted to the fact that each result given here is stated only for functions that are real-valued, and that most of these results are true for more general functions; in fact, each result is true when Ω is replaced by an arbitrary compact, 2° countable topological space, with the exception of those results that make reference to the ordering of the points in R.

§1. CLUSTER SETS AND ESSENTIAL CLUSTER SETS.

Let f be a mapping from the real line R into the extended real line Ω, and let x be a point on R. Then the following ordinary and essential cluster sets of f will be used throughout:

(i) The <u>right cluster set</u> $C^+(f,x)$ <u>of</u> f <u>at</u> x is the set of all points $\omega \in \Omega$ for which $f^{-1}(U) \cap (x,x+r) \neq \emptyset$ for each $r > 0$ and each open neighborhood U of ω. The <u>left cluster set</u> $C^-(f,x)$ <u>of</u> f <u>at</u> x is defined analogously, and the <u>cluster set</u> <u>of</u> f <u>at</u> x is the set

$$C(f,x) = C^+(f,x) \cup C^-(f,x)$$

(ii) The <u>right essential cluster set</u> $C_e^+(f,x)$ <u>of</u> f <u>at</u> x is the set of all points $\omega \in \Omega$ for which $f^{-1}(U)$ has positive upper right exterior density at x for each open neighborhood U of ω. The <u>left essential cluster set</u> $C_e^-(f,x)$ is defined similarly, and the _essential cluster set of_ f _at_ x is the set

$$C_e(f,x) = C_e^+(f,x) \cup C_e^-(f,x)$$

(iii) $\omega \in M(f,x)$ iff $f^{-1}(U) \cap (x-r,x+r)$ has positive exterior measure for each $r > 0$ and each open neighborhood U of ω.

(iv) $\omega \in H(f,x)$ iff $f^{-1}(U)$ has upper exterior density 1 at x for each open neighborhood U of ω.

(v) $\omega \in HB(f,x)$ iff $f^{-1}(U)$ has exterior density 1 at x for each open neighborhood U of ω.

The sets $M(f,x)$ and $H(f,x)$ were recently introduced by L. Zajíček [26], and it seems that the set $HB(f,x)$ is being introduced for the first time right here. The following result gives a relationship between the sets (ii) - (iv):

THEOREM (Zajíček [26]): If $f: R \to \Omega$ _is arbitrary, then the set_ $\{x: H(f,x) \not\subset M(f,x)\}$, _and hence the set_

$$\{x: H(f,x) \not\subset C_e(f,x)\}\ ,$$

is of the first category on R.

Henry Blumberg established the following relationship between the sets (ii) and (v):

THEOREM (Blumberg [8]): If $f: R \to \Omega$ _is arbitrary, then the set_

$$\{x: HB(f,x) \not\subset C_e(f,x)\}$$

is of measure zero on R.

It is noted that there exists a measurable function $f: R \to \Omega$ for which the set $\{x: HB(f,x) \not\subset C_e(f,x)\}$ is residual on R: Let T be a residual subset of R having measure zero. Then, according to Casper Goffman [14] there exists a measurable subset S of R such that the density of S exists at no point of T. Hence, if f is the characteristic function of S, then $HB(f,x) = \emptyset$ for each $x \in T$, and f is the desired function.

§2. SYMMETRY THEOREMS.

In 1907, W. H. Young [22] proved the following theorem: *If $f: R \to \Omega$ is an arbitrary one- or many-valued function, then*

$$\sup C^-(f,x) = \sup C^+(f,x) \quad \text{and} \quad \inf C^-(f,x) = \inf C^+(f,x)$$

for all but countably many points $x \in R$. Referring to this result, Collingwood wrote [12, p.4]:

> This appears to have been the first theorem to be explicitly stated for arbitrary functions. Young announced it at the meeting of the British Association held at Leicester in 1907. He used to refer to it as the Leicester theorem. A year later, at the Rome congress of 1908, he announced what he called the Rome theorem [23].

YOUNG'S ROME THEOREM: *If $f: R \to \Omega$ is an arbitrary one- or many-valued function, then*

$$C^+(f,x) = C^-(f,x)$$

for all but countably many points $x \in R$.

In 1924, S. Kempisty established the first known relationship between the left and right essential cluster sets of an arbitrary function.

THEOREM (Kempisty [18]): *If $f: R \to \Omega$ is arbitrary, then*

$$\sup C_e^-(f,x) \geq \inf C_e^+(f,x) \quad \text{and} \quad \sup C_e^+(f,x) \geq \inf C_e^-(f,x)$$

for all but countably many points $x \in R$.

This result furnishes a partial essential cluster set analogue to the theorems of Young; some time later, Z. Zahorski posed the question as to whether the exact essential cluster set analogues are true. In 1960, L. Belowska answered this question in the negative.

THEOREM (Belowska [4]): *There exists a measurable function* $f: R \to \Omega$ *for which* $\sup C_e^+(f,x) < \sup C_e^-(f,x)$ *at uncountably many points* $x \in R$.

In the same year, M. Kulbacka showed that there is a limit to the "size" of the set of points x at which the inequality in Belowska's theorem can occur; this result also furnishes a nice essential cluster set analogue of Young's Rome theorem.

THEOREM (Kulbacka [20]): *If* $f: R \to \Omega$ *is arbitrary, then the set of points* $x \in R$ *at which* $C_e^+(f,x) \neq C_e^-(f,x)$ *is of the first category and measure zero on* R.

The proofs of Belowska and Kulbacka are rather lengthy, but Casper Goffman [15] has given short and simple proofs of both results.

Just recently, using the concept of σ-porosity which was first introduced by E. P. Dolzhenko [13] in his work on complex-valued functions defined in the open unit disk, L. Zajíček gave the following improvement of Kulbacka's result.

THEOREM (Zajíček [26]): *If* $f: R \to \Omega$ *is arbitrary, then the set of points* $x \in R$ *at which* $C_e^+(f,x) \neq C_e^-(f,x)$ *is a σ-porous set of the type* $F_{\sigma\delta\sigma}$.

(A set $P \subset R$ is _porous at the point_ $x \in R$ provided that

$$\limsup_{\varepsilon \to 0} \frac{\gamma(x,\varepsilon,P)}{\varepsilon} > 0 \quad ,$$

where $\gamma(x,\varepsilon,P)$ is the length of the largest open interval in the complement of P which is entirely contained in the interval $(x-\varepsilon,x+\varepsilon)$. Then P is a _porous_ set if it is porous at each of its points, and it is a σ-_porous_ set if it is the countable union of porous sets. Such a set is both of the first category and of measure zero, but not every set of measure zero is σ-porous.)

Zajíček has also proven the following result which contains the result of Kempisty mentioned above.

THEOREM (Zajíček [26]): If $f\colon R \to \Omega$ _is arbitrary, then the set of points_ x _at which_

$$C_e^+(f,x) \cap C_e^-(f,x) = \emptyset$$

is countable.

§3. MEMBERSHIP OF $f(x)$ IN THE CLUSTER SETS OF f AT x.

W. H. Young ([24] and [25]) was the first to establish a relationship between the values of $f(x)$ and the set $C(f,x)$. He proved: _If_ $f\colon R \to \Omega$ _is an arbitrary one- or many-valued function, then the set of points_ x _at which some value of_ $f(x)$ _does not satisfy_ $\inf C(f,x) \le f(x) \le \sup C(f,x)$ _is countable._ Some fifty years later, E. F. Collingwood gave the following improvement of this result.

THEOREM (Collingwood [12]): If $f\colon R \to \Omega$ _is an arbitrary one- or many-valued function, then the set of points_ x _at which some value of_ $f(x)$ _satisfies_ $f(x) \notin C(f,x)$ _is countable._

This says that at all but countably many points ξ each value of $f(\xi)$ is approached by $f(x)$ as $x \to \xi$ through some sequence of points. In 1923, Henry Blumberg proved that at almost every point ξ each value of $f(\xi)$ is approached by $f(x)$ as $x \to \xi$ through some set of points having exterior density 1 at ξ. In cluster set notation, this result is stated as follows.

THEOREM (Blumberg [6]): *If $f\colon R \to \Omega$ is an arbitrary one- or many-valued function, then the set of points x at which some value of $f(x)$ satisfies $f(x) \notin HB(f,x)$ is of measure zero.*

The characteristic function of any residual set of measure zero can be used to illustrate the fact that the exceptional set in the previous theorem can be residual even if f is assumed to be measurable and if $HB(f,x)$ is replaced by $M(f,x)$.

§4. BOUNDARY CLUSTER SETS.

Here an example is given to illustrate how an important cluster set result of E. F. Collingwood concerning functions f mapping the open upper half plane H into Ω can be proved using only cluster set results for functions $\hat{f}\colon R \to \Omega$.

Let $f\colon H \to \Omega$ be arbitrary. The *cluster set* $C(f,x)$ *of* f *at* $x \in R$ is the set of all points $\omega \in \Omega$ for which there exists a sequence of points $z_n \in H$ with $z_n \to x$ and $f(z_n) \to \omega$. Then the *right boundary cluster set* $C_{Br}(f,x)$ *of* f *at* x is:

$$C_{Br}(f,x) = \bigcap_{\varepsilon>o} \overline{\Lambda(x,\varepsilon)}$$

where $\Lambda(x,\varepsilon) = \bigcup_{x<y<x+\varepsilon} C(f,y)$, and the bar denotes closure. The left boundary cluster set $C_{B\ell}(f,x)$ is defined similarly.

THEOREM (Collingwood [11]): *If* $f\colon H \to \Omega$ *is arbitrary, then*

$$C_{Br}(f,x) = C_{B\ell}(f,x) = C(f,x)$$

for all but countably many points $x \in R$.

Proof. Define the (many-valued) function $\hat{f}\colon R \to \Omega$ by

$$\hat{f}(x) = C(f,x)$$

Then it is easy to see that

$$C^{+}(\hat{f},x) = C_{Br}(f,x) \quad \text{and} \quad C^{-}(\hat{f},x) = C_{B\ell}(f,x)$$

According to the theorem of Collingwood cited in section 3, there exists a countable subset A of R such that, for every $x \in R-A$, each value of $\hat{f}(x)$ is an element of $C(\hat{f},x)$, that is,

(1) $$C(f,x) \subset C(\hat{f},x) \qquad (x \in R-A)$$

Also, according to Young's Rome theorem, there exists a countable subset B of R such that

$$C^{+}(\hat{f},x) = C^{-}(\hat{f},x) = C(\hat{f},x)$$

for each $x \in R-B$. That is,

(2) $$C_{Br}(f,x) = C_{B\ell}(f,x) = C(\hat{f},x) \qquad (x \in R-B)$$

Then (1) and (2) combined with the trivial fact that $C_{Br}(f,x)$ and $C_{B\ell}(f,x)$ are subsets of $C(f,x)$ for each x yields

$$C_{Br}(f,x) = C_{B\ell}(f,x) = C(f,x)$$

for each $x \in R - (A \cup B)$, and the proof is complete.

§5. AMBIGUITY THEOREMS.

A subset γ of H is called an *arc at* $x \in R$ if γ is a Jordan arc in H having one endpoint at x and the other endpoint at a point of H. The *cluster set* $C_{\gamma}(f,x)$ *of* $f\colon H \to \Omega$ *along an arc* γ *at* x is the set of all points $\omega \in \Omega$ for which there exists a sequence of points z_n on γ with $z_n \to x$ and $f(z_n) \to \omega$. Should γ be a rectilinear segment emanating from x and making the angle θ with the positive real line, the cluster set of f along γ is denoted by $C(f,x,\theta)$.

The function $f: H \to R$ is said to be _ambiguous at_ $x \in R$ if there exist two arcs α and β at x with

$$C_\alpha(f,x) \cap C_\beta(f,x) = \emptyset \quad .$$

Also f is said to be _rectilinearly ambiguous at_ x if there exist two directions θ_1 and θ_2 with

$$C(f,x,\theta_1) \cap C(f,x,\theta_2) = \emptyset \quad .$$

In 1930, Henry Blumberg proved the first theorem dealing with the set of points at which a function is rectilinearly ambiguous.

THEOREM (Blumberg [7]): Let $f: H \to \Omega$ _be arbitrary, and let_ θ_1 _and_ θ_2 _be two fixed directions. Then the set of points_ x _at which_ $\sup C(f,x,\theta_1) < \inf C(f,x,\theta_2)$ _is countable._

Four years later, Mabel Schmeiser gave the following improvement of this result.

THEOREM (Schmeiser [21]): If $f: H \to \Omega$ _is arbitrary, then the set of points_ $x \in R$ _for which there exist two directions_ θ_1 _and_ θ_2 _with_

$$\sup C(f,x,\theta_1) < \inf C(f,x,\theta_2)$$

is countable.

Then, in 1936, Vojtěch Jarník completely determined the nature of the set of points at which an arbitrary function is rectilinearly ambiguous.

THEOREM (Jarník [17]): *An arbitrary function* $f: H \to \Omega$ *is rectilinearly ambiguous at only countably many points* $x \in R$.

Finally, in 1955, modifying an idea due to Blumberg [9, pp. 16-17], Frederick Bagemihl [1] gave the final answer to the general ambiguity question.

BAGEMIHL'S AMBIGUOUS POINT THEOREM: *An arbitrary function* $f: H \to \Omega$ *can be ambiguous at only countably many points* $x \in R$.

With this settled, it is natural to ask what happens in the case of three arcs or three segments. To begin with, in his 1936 paper, Jarník had already constructed a function $f: H \to \Omega$ having the property that

$$C(f,x,\pi/4) \cap C(f,x,\pi/2) \cap C(f,x,3\pi/4) = \emptyset$$

for each irrational point x in Cantor's middle-third set. Then, in 1959, F. Bagemihl, G. Piranian, and G. S. Young [2] established the existence of a function $f: H \to \Omega$ with the property that to each $x \in R$ there correspond directions θ_1, θ_2, θ_3 for which

$$(*) \qquad C(f,x,\theta_1) \cap C(f,x,\theta_2) \cap C(f,x,\theta_3) = \emptyset \quad ,$$

that is, f has the 3-*segment property at each* $x \in R$.

Now suppose $f: H \to \Omega$ is continuous. Then $C(f,x,\theta)$ is connected for each x and each θ; hence, the set of points x at which f has the 3-segment property is countable since (*) implies that f is rectilinearly ambiguous at x. It is noted that the case for complex-valued functions is not as easily resolved. In their 1959 paper, Bagemihl, Piranian, and Young posed the following question: *Does there exist a continuous complex-*

valued function f in H having the 3-segment property at each point of a set of positive measure or of second category on R? This question has never been answered.

It remains to discuss the ambiguity theorems for essential directional cluster sets; the first of which was established by A. M. Bruckner and C. Goffman in 1966. (The symbol $C_e(f,x,\theta)$ is used to denote the essential cluster set of f at x in the direction θ.)

THEOREM (Bruckner-Goffman [10]): Let $f\colon H \to \Omega$ be continuous and let θ_1 and θ_2 be directions. Then the set of points x at which

$$\sup C_e(f,x,\theta_1) < \inf C_e(f,x,\theta_2)$$

is of the first category on R.

In their 1968 paper, C. Goffman and W. T. Sledd proved an interesting result (Theorem 2) of which the following generalization and extension of the Bruckner-Goffman theorem is a immediate consequence.

THEOREM (Goffman-Sledd [16]): Let $\theta_1,\theta_2,\ldots$ be a sequence of directions. If $f\colon H \to \Omega$ is measurable, then the set of points x at which

$$\bigcap_{n=1}^{\infty} C_e(f,x,\theta_n) = \emptyset$$

is of measure zero; furthermore, if f is continuous, then the set of such points x is of the first category.

Goffman and Sledd also pointed out the fact that "continuous" can not be replaced by "measurable" in the statement of the previous theorem as shown by the characteristic function f of the set $S \times R^+$, where R^+ is the positive real line and S is any residual subset of R of measure zero. For such a function f,

$$C_e(f,x,\pi/2) \cap C_e(f,x,\theta) = \emptyset$$

for each $x \in S$ and each $\theta \neq \pi/2$.

Using a construction of J. R. Kinney [19], the present author [3] proved the existence of a measurable function $f: H \to \Omega$ having the property that, at almost every $x \in R$, there exists a direction θ_x for which

$$C_e(f,x,\pi/2) \cap C_e(f,x,\theta_x) = \emptyset \quad .$$

In view of this example and the example given in the previous paragraph, it is clear that "no" essential cluster set analogue of Jarník's theorem exists.

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