Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 281–291

Claude Tricot, Département de mathématiques appliquées, École Polytechnique, P.O. Box 6079 - station A, Montréal (Québec) - Canada, H3C 3A7. e-mail: claude_tricot@mathappl.polymtl.ca

Curves, length, fractal dimension

1. Introduction

For the sake of simplicity, we limit our attention to bounded, simple curves in the plane. Our main goal is to relate the study of finite-length (rectifiable) curves to that of infinite-length (fractal) curves. As usual, this relationship can be established with the help of the notion of orders of growth for functions (to 0 or $+\infty$). For a parametrized curve, the length (in the rectifiable case) is, in general, the integral of the velocity but this approach is not easily adapted to fractal curves. The parameter induces, in both cases, a local analysis which gives deep insight into the structure of the curve.

Notation *$[a,b]$ is the parameter interval ($a < b$), t the parameter, $\gamma : [a,b] \rightarrow \mathbb{R}^2$ a continuous, one-to-one function (the parametrization), $\Gamma = \gamma([a,b])$ the curve, and for any $a < t_1 < t_2 < b$, $\gamma(t_1) \frown \gamma(t_2) = \gamma([t_1,t_2])$ is an arc of Γ, of measure $t_2 - t_1$. The Hausdorff distance between two compact sets E_1, E_2 of the plane is denoted $dist(E_1, E_2)$.*

2. Relating velocity and length

For any sequence (P_n) of polygonal curves, having same endpoints as Γ, whose vertices belong to Γ, and such that $\mathrm{dist}(P_n, \Gamma) \rightarrow 0$, the length of Γ is defined as

$$L(\Gamma) = \lim_n L(P_n) = \sup_n L(P_n). \tag{1}$$

If γ is continuously differentiable over $[a,b]$, there exists at all $x = \gamma(t)$ of Γ a *speed* $v(t)$, defined as

$$v(t) = \lim_{\tau \rightarrow 0} \frac{\mathrm{dist}(\gamma(t-\tau), \gamma(t+\tau))}{2\tau}. \tag{2}$$

Then, the formula

$$(3) \qquad L(\Gamma) = \int_a^b v(t)dt$$

is obtained through a standard exchange limit-integral. Formula (3) is no longer valid when $v(t)$ does not exist everywhere; existence *almost everywhere* on $[a, b]$ is not sufficient.

Example 1 *The devil's staircase (figure 1) is the graph of an increasing, continuous function. Using the abcissa as the parameter, one sees that* $v(t)$ *exists (and is equal to 1) everywhere in* $[0, 1]$ *but on the Cantor set* C*. We get* $\int_{[0,1]\setminus C} v(t)dt = 1$*, but* $L(\Gamma) = 2$.

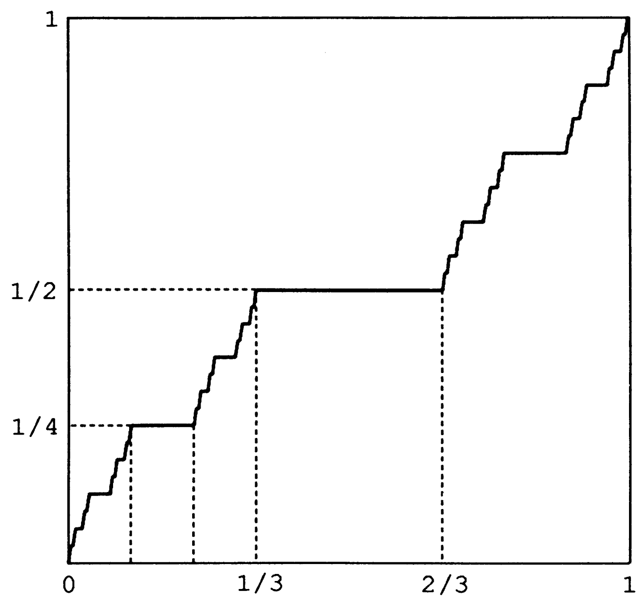

Figure 1: The devil's staircase

It is possible to transform (3) into a general formula for the length, by avoiding to take a limit in (2). First, define the two-variables function

$$d(t, \tau) = \begin{cases} dist(\gamma(a), \gamma(a + 2\tau)) & \text{if } t - \tau \leq a \\ dist(\gamma(t - \tau), \gamma(t + \tau)) & \text{if } a \leq t - \tau < t + \tau \leq b \\ dist(\gamma(b - 2\tau), \gamma(b)) & \text{if } b \leq t + \tau. \end{cases}$$

Then $\bar{d}_\tau = \dfrac{1}{b-a}\displaystyle\int_a^b d(t,\tau)dt$ is the average distance run in time 2τ, and $\bar{d}_\tau/2\tau$ is an *averaged local velocity*.

Theorem 1 *For any simple, bounded curve* Γ,

$$L(\Gamma) = (b-a)\lim_{\tau\to 0}\frac{\bar{d}_\tau}{2\tau} \tag{4}$$

(length = time × velocity).

The proof uses the relationship between Riemann sums and the integral.

3. Orders of growth and fractal dimension

Let $\Gamma(\epsilon) = \{x \in \mathbb{R}^2 \mid \operatorname{dist}(x,\Gamma) \le \epsilon\}$, and $\mathcal{A}_\epsilon$ be the area of $\Gamma(\epsilon)$. When Γ has finite length, $L(\Gamma)$ may be obtained by

$$L(\Gamma) = \lim_{\epsilon\to 0}\frac{\mathcal{A}_\epsilon}{2\epsilon} \qquad \text{(Minkowski's definition of length).} \tag{5}$$

The curve Γ has infinite length if

$$\overline{\lim}_{\epsilon\to 0}\frac{\mathcal{A}_\epsilon}{\epsilon} = +\infty \qquad (\text{ in this case } \underline{\lim}_{\epsilon\to 0}\frac{\mathcal{A}_\epsilon}{\epsilon} \text{ is also } +\infty).$$

Using a way of thinking inspired by E. Borel, F. Hausdorff, G. Bouligand and others, one tries to characterize Γ by the *order of growth* of the function $\mathcal{A}_\epsilon$ when ϵ decreases to 0. When the family of power functions t^α, $\alpha > 0$, is the reference scale of functions, the order of growth of $\mathcal{A}_\epsilon$ is the limit (when it exists) of $\log\mathcal{A}_\epsilon/\log\epsilon$. In order to get a limit which increases from 1 to 2 when Γ becomes more and more *irregular* ($\mathcal{A}_\epsilon$ is slower to reach 0), one defines the *fractal dimension* of Γ as

$$\Delta(\Gamma) = \lim_{\epsilon\to 0}\left(2 - \frac{\log\mathcal{A}_\epsilon}{\log\epsilon}\right). \tag{6}$$

Use $\overline{\lim}$ when this does not converge. For the sake of simplicity, we will assume throughout this paper that $\Delta(\Gamma)$ is always a limit.

For practical applications, the fractal dimension is difficult to handle, and it would be interesting to relate it to the sequences $(L(P_n))$, as in (1), or to the local velocity, as in (4). This is what we will investigate now.

4. How to relate $\Delta(\Gamma)$ and $(L(P_n))$?

Following Section 2, let us define P_ϵ a polygonal curve, with same endpoints as Γ, whose vertices belong to Γ, and such that every steplength (except perhaps the last one) is equal to ϵ. Take $L_\epsilon = L(P_\epsilon)$. A reasonable guess (the one which was made by almost everyone during the 80's!) is the following:

$$\Delta(\Gamma) = \lim_{\epsilon \to 0} \left(1 + \frac{\log L_\epsilon}{|\log \epsilon|}\right). \tag{7}$$

This formula is only true for self-similar curves (see Section 2). In general, it is a *false guess*, as proven by the two following examples:

Example 2 *Take the attractor of an iterated function system, as in Figure 2. This is a well-known curve whose fractal dimension is $\frac{3}{2}$. By construction, a sequence of approximating polygons arises directly:*

- *P_0 is a segment, of length $\sqrt{2}$.*
- *P_n is made up with 4^n segments of length $\epsilon_n = 2^{-n}\sqrt{1+4^{-n}}$.*

Then using $L(P_n) = \sqrt{1+4^n} \simeq 2^n$, one gets

$$1 + \frac{\log L(P_n)}{|\log \epsilon_n|} \to 2.$$

Example 3 *Take a parameter $b < \frac{1}{4}$. Inside an original isocel triangle, of basis 1, of height h, construct a chain of 6 isocel triangles of basis $1/4$, of height hb and so on. The n-th step consists in 6^n isocel triangles (embedded in the previous ones), of basis 4^{-n}, of height hb^n. With $L(P_n) = 6^n 4^{-n} = (3/2)^n$, and $\epsilon_n = 4^{-n}$, one gets*

$$1 + \frac{\log L(P_n)}{|\log \epsilon_n|} = 1 + \frac{\log 3/2}{\log 4}.$$

On the other hand,

$$\Delta(\Gamma) = 1 + \frac{\log 3/2}{|\log b|}$$

depends actually on the parameter b. The self-similar case is obtained for $b = 1/4$; the two formulas coincide only in this case.

It is easy to see why formula (7) is a false guess in these examples: in (7), we should replace ϵ (the steplength of P_ϵ) by the Hausdorff distance $\mathrm{dist}(P_\epsilon, \Gamma)$, in order to get the "good" result. The Hausdorff distance between the two curves must be, locally, the same all along Γ. This may help to understand the following sections.

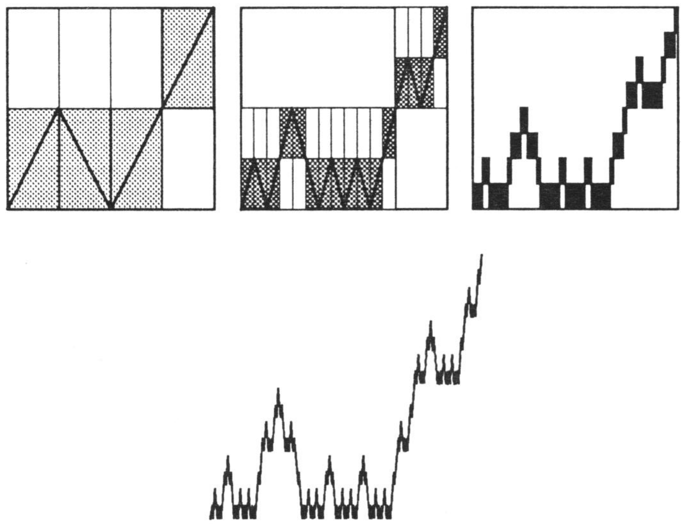

Figure 2: First steps of the construction of a self-affine function

5. Expansive curves

Let $\mathcal{H}$ denote the set of all compact sets in $\mathbb{R}^2, and, forallE \in \mathcal{H}, \mathrm{diam}(E)$ denote its diameter, $K(E)$ its convex hull and $b(E)$ the breadth of $K(E)$.

A *size function*, defined on $\mathcal{H}$, has the 3 following properties:

1. There exists 2 constants $0 < c_1 < c_2$ such that
$$c_1 \,\mathrm{diam}(E) \leq \mathrm{size}(E) \leq c_2 \,\mathrm{diam}(E);$$
2. $E_1 \subset E_2 \Rightarrow \mathrm{size}(E_1) \leq \mathrm{size}(E_2)$;
3. $\lim\limits_{n\to\infty} \mathrm{dist}(E_n, E) = 0 \Leftrightarrow \lim\limits_{n\to 0} \mathrm{size}(E_n) = \mathrm{size}(E)$.

Example 4 *The diameter, the perimeter of $K(E)$, the diameter of the circumscribed circle, are all size functions (for different constants c_1 and c_2).*

A *deviation function*, defined on $\mathcal{H}$, has the 3 following properties:

1. There exists 2 constants $0 < c_1 < c_2$ such that
$$c_1 b(E) \leq \mathrm{dev}(E) \leq c_2 b(E);$$

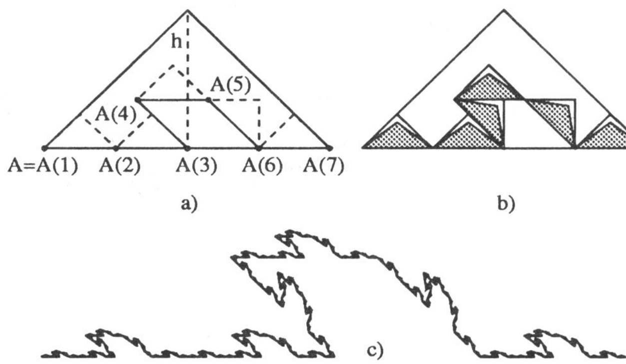

Figure 3: Construction of a 6-sided polygonal curve inside an isocel triangle

2. $E_1 \subset E_2 \Rightarrow \text{dev}(E_1) \leq \text{dev}(E_2)$;

3. $\lim_{n\to\infty} \text{dist}(E_n, E) = 0 \Leftrightarrow \lim_{n\to 0} \text{dev}(E_n) = \text{dev}(E)$.

Example 5 *The breadth of $K(E)$, the inner diameter of $K(E)$, the Hausdorff distance of E to a diameter of E, are all deviation functions.*

Proposition 1 *For any size and deviation functions,*

$$\mathcal{A}(K(E)) \simeq \text{size}(E)\,\text{dev}(E).$$

These two set functions may be chosen according to the geometry of the problem. We will now assume that a choice has been made.

Proposition 2 *For any curve Γ, and for any $\epsilon < \text{dev}(\Gamma)$, one can define a covering $(\Gamma_i^\epsilon)_{1\leq i\leq N}$ of Γ by arcs of deviation ϵ, such that*

$$\forall i,\ 1 \leq i \leq N-2,\quad \begin{aligned} \Gamma_i^\epsilon \cap \Gamma_{i+2}^\epsilon &= \emptyset \\ \Gamma_i^\epsilon \cap \Gamma_{i+1}^\epsilon &= \{A_{i+1}\} \end{aligned}$$

where A_{i+1} is the only common point between Γ_i^ϵ and Γ_{i+1}^ϵ.

Definition 1 *A curve Γ is* expansive *if there exists a constant $c > 1$, and, for all $\epsilon < \text{dev}(\Gamma)$, a covering (Γ_i^ϵ) of Γ, such that*

(i) $\forall i,\ \dfrac{\epsilon}{c} \leq \text{dev}(\Gamma_i^\epsilon) \leq \epsilon$

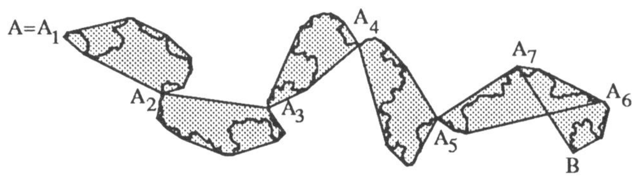

Figure 4: A uniform deviation procedure along the curve

(ii) $\sum_i \mathcal{A}(K(\Gamma_i^\epsilon)) \leq c\mathcal{A}(\bigcup_i K(\Gamma_i^\epsilon))$.

Property (i) indicates that the covering of Γ is a *uniform deviation procedure* along the curve. Property (ii) indicates that the local convex hulls do not overlap too much; intuitively, at any scale the curve does not come back on itself.

6. Uniform deviation procedures on Γ

Now we can relate the fractal dimension $\Delta(\Gamma)$ to the order of growth of some sequence $(L(P_\epsilon))$; the condition is that the steps of P_ϵ have constant deviation. Let us recall that, for every covering (Γ_i) of a curve Γ, the *index* of (Γ_i) is the largest of all integers n such that there exists n arcs in (Γ_i) whose intersection contains more that one point.

Theorem 2 *Let Γ it be an expansive curve, c be a constant and ω an integer, both greater than 1. For every $\epsilon < \mathrm{dev}(\Gamma)$, let (Γ_i^ϵ) be a covering of Γ whose index is $\leq \omega$, and for all i,*

$$\frac{\epsilon}{c} \leq \mathrm{dev}(\Gamma_i^\epsilon) \leq \epsilon.$$

Then

$$\Delta(\Gamma) = \lim_{\epsilon \to 0} \left(1 + \frac{\log \sum_i \mathrm{diam}(\Gamma_i^\epsilon)}{|\log \epsilon|} \right). \tag{8}$$

The main argument of the proof consists in showing that $\sum \mathcal{A}(K(\Gamma_i^\epsilon))$ is equivalent to $\mathcal{A}(\Gamma(\epsilon))$.

7. Examples

In Example 2 of Section 5., at stage n, Γ is covered by 4^n arcs whose deviation has the order 4^{-n} (breadth of the fundamental rectangles covering Γ), each having size $\simeq 2^{-n}$. Take $\epsilon_n = 4^{-n}$, $\sum_i \operatorname{diam}(\Gamma_i^{\epsilon_n}) \simeq 2^n$, to get

$$\Delta(\Gamma) = 1 + \frac{\log 2}{\log 4} = \frac{3}{2}.$$

In Example 3 of Section 4, Γ is covered by 6^n arcs of size $\simeq 4^{-n}$, deviation $\simeq b^n$. Then

$$\Delta(\Gamma) = 1 + \frac{\log 6/4}{|\log b|}.$$

This converges to 1 as $b \to 0$, and to $\dfrac{\log 6}{\log 4}$ as $b \to 1/4$.

8. Applications

There are two main applications concerning fractal curves:

- Let τ be the graph of a nowhere differentiable function $z(t)$, $t \in [0, 1]$. We assume that $z(t)$ is, uniformly on $[0, 1]$, holderian with exponent H, $0 < H < 1$. Let us, moreover, assume (as in Example 1) that, for all t and $\tau \in [0, 1]$, there exists t_1 and t_2 in $[t-\tau, t+\tau]$, such that $z(t_1) = z(t_2)$, and $|t_1 - t_2| \geq c\tau$ for some constant c. Then, the subarc of Γ of extremities $(t-\tau, z(t-\tau))$ and $(t+\tau, z(t+\tau))$ has deviation $\simeq \tau$ and size $\simeq \tau^H$. With $\epsilon = \tau$ and $\sum_i \operatorname{diam} \Gamma_i^\epsilon \simeq \frac{1}{\epsilon}\epsilon^H$, Theorem 2 gives

  $$\Delta(\Gamma) = 2 - H,$$

  a well-known result for holderian functions.

- Let Γ be a strictly self-similar curve, made up with N similarities of ratio ρ, $0 < \rho < 1$. For any n, Γ can be covered by N^n similar arcs, of size $\simeq$ deviation $\simeq \rho^n$. Then, Theorem 2 gives

  $$\Delta(\Gamma) = 1 + \frac{\log N\rho}{|\log \rho|} = \frac{\log N}{|\log \rho|},$$

  again a well-known result. All the curves of Section 7 and Section 8 are expansive.

9. Uniform deviation curves

All previous examples belong to the following class of curves:

Definition 2 *Let Γ be a parametrized curve (notations in Section 1). We say that Γ has "uniform deviation" if there exists a function $g(\tau)$ such that, for all $\tau < b - a$, every $x \in \Gamma$ belongs to an arc of measure τ, and deviation $\simeq g(\tau)$.*

We can get the fractal dimension of such curves, when $g(\tau) = \tau^\beta$:

Theorem 3 *Let Γ be an expansive curve, of uniform deviation with $g(\tau) = \tau^\beta$. Let $T(t, \tau)$ be the size of the arc $\gamma(t - \tau) \frown \gamma(t + \tau)$, and*

$$\overline{T}_\tau = \frac{1}{b-a} \int_a^b T(t, \tau) dt$$

be the average local size. Then

$$\Delta(\Gamma) = 1 + \frac{1}{\beta} - \frac{1}{\beta} \lim_{\tau \to 0} \frac{\log \overline{T}_\tau}{\log \tau}. \tag{9}$$

Many mathematical models of curve have uniform deviation, because of the homogeneity of their construction. However, attractors of iterated affine functions systems *do not* have this property in general, at least not with respect to the parametrization induced by the construction. One question is in order here:

> *Is it possible, for every simple curve Γ, to define a parametrization such that Γ has uniform deviation?*

10. The (α, β) characterization of curves

Formula (9) becomes a nice, compact formula when $\overline{T}_\tau \simeq \tau^\alpha$, for some parameter $\alpha > 0$. Then

$$\Delta(\Gamma) = \frac{\beta + 1 - \alpha}{\beta}, \tag{10}$$

a formula which shows the relationship between fractal dimension and the two indices characterizing the local geometry of the parametrized curve Γ: α gives the average local size, and β gives the local deviation. It is not too difficult to show that the pair (α, β) must follow the three following conditions:

$$0 < \alpha \leq 1, \quad \alpha \leq \beta, \quad \alpha + \beta > 1.$$

(use the continuity of γ, the inequality $\operatorname{dev} E \leq c \operatorname{diam} E$ for some constant c, and the expansivity of Γ).

Figure 5 shows a part of the 2-dimensional graph of Δ, as a function of α and β. One can recognize the two main cases of Section 8: $\beta = 1$, which gives $\Delta(\Gamma) = 2 - \alpha$, α having the meaning of an Hölder exponant; and $\beta = \alpha$ which gives $\Delta = \dfrac{1}{\alpha}$. To the last case belong all strictly self-similar curves. Conversely, we could define the non-strict (or statistical) self-similarity as follows:

Definition 3 *A parametrized curve Γ is statistically self-similar if it has uniform deviation, and $\alpha = \beta$.*

When $\alpha = 1$, then $\Delta(\Gamma) = 1$: rectifiable curves belong to this case. When a curve is twice continuously differentiable, then $\alpha = 1$ et $\beta = 2$.

Problem *Construct a curve such that $\alpha = 1$ and β is any parameter > 1.*

11. Fractal dimension, and the local velocity

For a fractal curve, whose every subarc has infinite length, there is no such thing as speed (Section 2): such a curve is the trajectory of an object having infinite speed. But it is possible to use a local velocity, as in Section 2. It is only better to replace the distance $d(t, \tau)$ between the two endpoints of an arc of measure 2τ, by its size. Keeping in mind the principle of uniform deviation, we define, for every $t \in [a, b]$, and for every $\epsilon < \operatorname{dev}(\Gamma)$, an arc $\Gamma(t, \epsilon)$ of deviation ϵ, containing the point $\gamma(t)$ of Γ (this may be done in several ways). Denote by $\mu(\Gamma(t, \epsilon))$ the measure of this arc, that is, the time passed in $\Gamma(t, \epsilon)$. Now to define a *local speed* we use the ratio

$$\frac{\operatorname{size}(\Gamma(t, \epsilon))}{\mu(\Gamma(t, \epsilon))}.$$

An approximate ϵ-length of Γ is given by the integral

$$\int_a^b \frac{\operatorname{size}}{\mu}\, dt.$$

We obtain the following generalization of (9):

$$\Delta(\Gamma) = \lim_{\epsilon \to 0} \left(1 + \frac{\log \displaystyle\int_a^b \frac{\operatorname{size}}{\mu} dt}{|\log \epsilon|} \right), \tag{11}$$

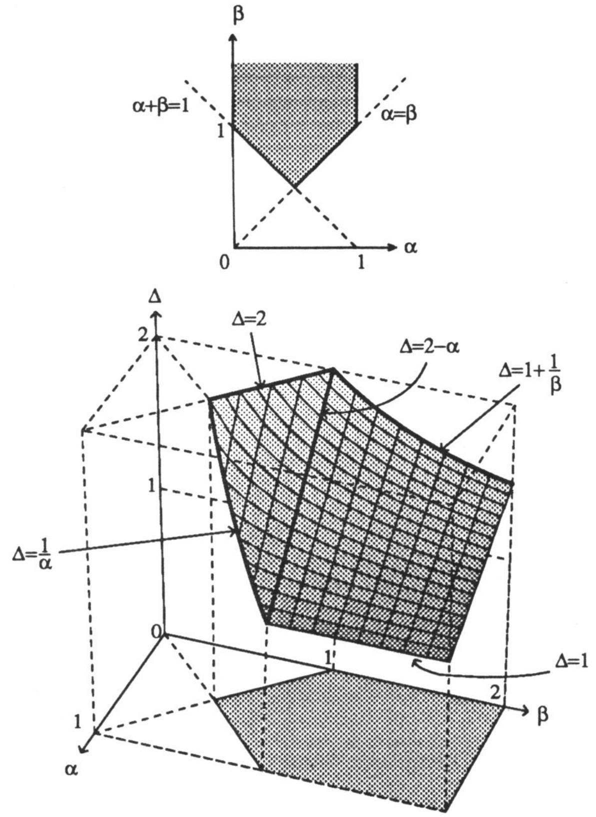

Figure 5: The 2-variable function $\Delta = \frac{\beta+1-\alpha}{\beta}$

but this is just a conjectural formula, equivalent to (9) in the case of uniform deviation ($\mu(\Gamma(t,\epsilon)) \simeq \epsilon^{1/\beta}$). It would be interesting to extend the validity of (11).

References

This is a survey of a part of the book "Curves and fractal dimension", to be published by *Springer-Verlag* (French and English versions).