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Does Every Borel Function Have a Somewhere Continuous Modification?

Is it possible to place any limits on the roughness of a Borel measurable function from the set of real numbers into itself? One answer to this question is provided by Lusin's theorem (see, for example, §8.29 of [9]): if $f : \mathbb{R} \to \mathbb{R}$ is Borel, then for every $\epsilon > 0$ there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\lambda(\{x : f(x) \neq g(x)\}) < \epsilon$, where λ is Lebesgue measure. In some sense, Lusin's theorem is the best possible result of this type. It is certainly not the case that any Borel function is equal almost everywhere to a continuous function, as shown by the trivial example

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What happens, however, if we look in a different direction and ask whether an arbitrary Borel function is almost everywhere equal to a function that is continuous at at least one point? That the answer to this question is negative is implied by a striking example of Carathéodory ([5], §§427 - 428) that exhibits a Borel function with the following remarkable property.

Definition. *A Borel function $f : \mathbb{R} \to \mathbb{R}$ has the property (Car) if for every non-empty, open set $A \subset \mathbb{R}$ and every non-empty, open set $B \subset \mathbb{R}$, the set $f^{-1}(A) \cap B$ has non-zero Lebesgue measure.*

It is clear that if f is any function with property (Car) and x is an arbitrary point in $\mathbb{R}$, then there is no function g such that $f = g$ almost everywhere and g is continuous at x.

Carathéodory's construction is quite involved and uses some relatively deep facts about singularities of functions of a complex variable. Berman [2] offers another, somewhat simpler, construction of a function with the property (Car). Indeed, Berman shows that his example displays the following, even more erratic, behaviour.

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Definition. *A Borel function $f : \mathbb{R} \to \mathbb{R}$ has the property (Ber) if for every set $A \subset \mathbb{R}$ with non-zero Lebesgue measure and every non-empty, open set $B \subset \mathbb{R}$, the set $f^{-1}(A) \cap B$ has non-zero Lebesgue measure.*

Unfortunately, Berman's example is a little unsatisfying in that he does not construct such a function explicitly. Instead, he finds a Gaussian stochastic process such that with probability one the sample paths of the process will be functions with property (Ber). We are grateful to the referee for pointing out that Carathéodory's example also satisfies (Ber), although this is not explicitly stated in [5]. Our aim here is to give a simple, concrete example of a function with property (Ber), and hence with property (Car).

Let (Ber*) be the analogue of property (Ber) for Borel functions from $[0,1[$ into $\mathbb{R}$; and note that if we are able to construct a function with property (Ber*), then the periodic continuation of such a function to $\mathbb{R}$ will have property (Ber).

For $n \in \mathbb{N}$, define a function $\epsilon_n : [0,1[\to \mathbb{R}$ by

$$\epsilon_n(x) = \begin{cases} +1, & \text{if } 2k/2^n \le x < (2k+1)/2^n, \quad k = 0, 1, \dots, 2^{n-1} - 1, \\ -1, & \text{if } (2k+1)/2^n \le x < (2k+2)/2^n, \quad k = 0, 1, \dots, 2^{n-1} - 1. \end{cases}$$

The function ϵ_n, or some minor variant thereof, is usually called the *n-th Rademacher function.* It is well-known that if we equip the interval $[0,1[$ with Lebesgue measure, then the Rademacher functions form a sequence of independent, identically distributed random variables with common distribution

$$\lambda\{\epsilon_n = +1\} = \lambda\{\epsilon_n = -1\} = 1/2,$$

(see, for example, [8] or Ch1 of [3] for a detailed discussion of this observation and its consequences). As remarked by Feller ([6], §V.3a), this fact "... has been utilized since the beginnings of probability theory."

Given a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, it follows from Kolmogorov's "three series" criterion for the convergence of a sum of independent random variables (see, for example, Theorem IX.9.3 of [6]) that the sum $\sum_{n=1}^{\infty} a_n \epsilon_n$ converges almost everywhere if and only if the sum $\sum_{n=1}^{\infty} a_n^2$ converges (see also §2.5 of [8] for a more elementary proof due to Paley and Zygmund and some comments on Rademacher's original proof). In particular, the sum $\sum_{n=1}^{\infty} n^{-1} \epsilon_n$ converges almost everywhere. Define $f : [0,1[\to \mathbb{R}$ by

$$f(x) = \begin{cases} 2\pi \sum_{n=1}^{\infty} n^{-1} \epsilon_n(x), & \text{if } \sum_{n=1}^{\infty} n^{-1} \epsilon_n(x) \text{ converges}, \\ 0, & \text{otherwise}. \end{cases}$$

Claim. *The function f has the property (Ber*).*

To see why this is so, let us first look at the following statement, which is the simplest instance of the sort of thing we have to prove.

Subclaim. *For all Borel sets $A \subset \mathbb{R}$ with $\lambda(A) > 0$, we have $\lambda(f^{-1}(A)) > 0$.*

Translated into probabilistic language, the subclaim states that Lebesgue measure is absolutely continuous with respect to the distribution of the random variable f. The idea we use for the proof is found in [2] and hinges on the following lemma. For completeness, we sketch the proof. We follow the convention that the characteristic function (that is, Fourier-Stieltjes transform) of a Borel probability measure μ on $\mathbb{R}$ is defined as $\hat{\mu}(y) = \int \exp(iyx)\mu(dx)$, $y \in \mathbb{R}$.

Lemma. *Let μ be a Borel probability measure on $\mathbb{R}$ with characteristic function $\hat{\mu}$ satisfying*

$$\int \exp(b|y|)|\hat{\mu}(y)|\,dy < \infty$$

for some $b > 0$. Then μ is absolutely continuous with respect to λ. Moreover, there is a function h defined on the strip $\{s + it \in \mathbb{C} : |t| < b\}$ and analytic in that set, such that the restriction of h to the real axis is the Radon-Nikodym derivative $d\mu/d\lambda$. Consequently, λ is also absolutely continuous with respect to μ.

Proof. As $\int |\hat{\mu}(y)|\,dy < \infty$, it follows by Fourier inversion that μ is absolutely continuous with respect to λ, with Radon-Nikodym derivative given by

$$\frac{d\mu}{d\lambda}(z) = (2\pi)^{-1} \int \exp(-izy)\hat{\mu}(y)\,dy, \qquad z \in \mathbb{R}.$$

By hypothesis, the above integral also exists for any $z \in S = \{s+it \in \mathbb{C} : |t| < b\}$. Let $h(z)$ denote the value of the integral. It follows easily from Lebesgue's dominated convergence theorem and our hypothesis that the function $z \mapsto h(z)$ is complex differentiable, and hence analytic, in S.

To show that λ is absolutely continuous with respect μ and complete the proof of the lemma, we need to show that $d\mu/d\lambda > 0$, λ-almost everywhere. This, however, follows immediately from the fact that the set of zeroes of an analytic function that is not identically zero is a set without limit points (see, for example, §4.3.2 of [1]), and such a set is countable . □

We can now complete the proof of the subclaim. The characteristic function of the distribution of the random variable ϵ_n is

$$y \mapsto \int_{[0,1[} \exp(iy\epsilon_n(x))\lambda(dx) = \exp(iy).\frac{1}{2} + \exp(-iy).\frac{1}{2} = \cos(y);$$

and hence, by the independence of the sequence $\{\epsilon_n\}_{n=1}^{\infty}$, the characteristic function of the distribution of the random variable f is

$$y \mapsto \prod_{n=1}^{\infty} \cos(2\pi y/n).$$

Observe that $|\cos(2\pi y/n)| \leq \frac{1}{2}$ whenever $4|y| \leq n \leq 6|y|$. As the number of integers lying in the interval $[4|y|, 6|y|]$ is at least $2|y| - 2$, we certainly have $|\prod_{n=1}^{\infty} \cos(2\pi y/n)| \leq 2^{-(2|y|-2)}$. The subclaim now follows from the lemma.

The proof of the claim is similar. It suffices to show that for every dyadic interval of the form $[j/2^m, (j+1)/2^m[$, $m \in \mathbb{N}$, $0 \leq j \leq 2^m - 1$, and all Borel sets $A \subset \mathbb{R}$ with $\lambda(A) > 0$, we have $\lambda(f^{-1}(A) \cap [j/2^m, (j+1)/2^m[) > 0$. Translated into probabilistic language, it must be shown that Lebesgue measure is absolutely continuous with respect to the conditional distribution of the random variable f given the event

$$[j/2^m, (j+1)/2^m[= \{x : \epsilon_n(x) = \epsilon_n(j/2^m), 1 \leq n \leq m\}.$$

By the independence of the sequence $\{\epsilon_n\}_{n=1}^{\infty}$, this conditional distribution is the same as the unconditional distribution of the random variable

$$x \mapsto 2\pi[\sum_{n=1}^{m} n^{-1}\epsilon_n(j/2^m) + \sum_{n=m+1}^{\infty} n^{-1}\epsilon_n(x)].$$

The characteristic function of this common distribution is

$$y \mapsto [\prod_{n=1}^{m} \exp(iy2\pi\epsilon_n(j/2^m)/n)][\prod_{n=m+1}^{\infty} \cos(2\pi y/n)].$$

The required result now follows from the lemma, as before.

Remarks. i) There is an extensive body of literature on the general theme that smoothness of the measure $\lambda \circ g^{-1}$ for a function g implies roughness of g itself. We refer the reader to [7] for an excellent survey.

ii) In the course of the proof, we showed that the distribution of the random variable f is absolutely continuous with respect to Lebesgue measure. It is a consequence of Jessen and Wintner's "law of pure types" (see, for example, Theorem 3.26 of [4]) that the distribution of any random variable of the form $\sum_n a_n\epsilon_n$ is either purely atomic, continuous but singular with respect to Lebesgue measure, or absolutely continuous with respect to Lebesgue measure.

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