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On the Equivalence of two Convergence Theorems for the Henstock Integral

There are two well-known convergence theorems for the Henstock integral. Let $\{f_n\}$ be a sequence of Henstock integrable functions that converge pointwise to a function f on $[a,b]$ and let $F_n(x) = \int_a^x f_n$ for each n. One of the theorems has an easy proof and requires that the sequence $\{f_n\}$ be uniformly Henstock integrable on $[a,b]$. The other, usually written in the language of the Denjoy-Perron integral, has a more difficult proof and requires that the sequence $\{F_n\}$ be equicontinuous and equi ACG_* on $[a,b]$. It is not easy to compare the hypotheses of these two theorems. Two recent attempts ([1] and [2]) have been made. In [2], the term equi ACG^∇ is introduced. It is shown that $\{F_n\}$ is equi ACG^∇ on $[a,b]$ if and only if $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$. Furthermore, it is possible to prove that $\{F_n\}$ equi ACG_* on $[a,b]$ implies $\{F_n\}$ equi ACG^∇ on $[a,b]$. In [1], a new convergence theorem is proved and it is shown to include both of the standard convergence theorems as special cases. Here the necessary hypothesis is that $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$. The purpose of this paper is to prove that $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ if and only if $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$.

We will assume that the reader is familiar with the terminology of the Henstock integral. The relevant notation needed for the paper appears below. Let $f, F : [a,b] \to R$, let $E \subset [a,b]$, let δ be a positive function defined on $[a,b]$, and let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le q\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. Then

$f(\mathcal{P}) = \sum_{i=1}^{q} f(x_i)(d_i - c_i)$ denotes the Riemann sum of f associated with $\mathcal{P}$;

$F(\mathcal{P}) = \sum_{i=1}^{q} \big(F(d_i) - F(c_i)\big)$, where F will always be an indefinite integral;

CE denotes the complement of E;

$\rho(x, E)$ denotes the distance from x to E; and

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$\mathcal{P}$ is E–subordinate to δ means that $\mathcal{P}$ is subordinate to δ and each of the tags x_i is in E.

Definition 1 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$ and let $F_n(x) = \int_a^x f_n$ for each n. The sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ if for each $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f_n(\mathcal{P}) - F_n(\mathcal{P})| < \epsilon$ for all n whenever $\mathcal{P}$ is subordinate to δ.*

Simple Convergence Theorem 1 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges pointwise to f on $[a,b]$. If the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$, then f is Henstock integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Definition 2 *Let $\{F_n\}$ be a sequence of functions defined on $[a,b]$ and let $E \subset [a,b]$ be measurable.*
(a) The sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy on E if $\{F_n\}$ converges pointwise on E and if for each $\epsilon > 0$ there exist a positive function δ on E and a positive integer N such that $|F_n(\mathcal{P}) - F_m(\mathcal{P})| < \epsilon$ for all $m, n \geq N$ whenever $\mathcal{P}$ is E–subordinate to δ.
(b) The sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on E if E can be written as a countable union of measurable sets on each of which $\{F_n\}$ is $\mathcal{P}$–Cauchy.

Another Convergence Theorem 1 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges pointwise to f on $[a,b]$. If the sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$, then f is Henstock integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Lemma 1 *Suppose that $f : [a,b] \to R$ and let $Z \subset [a,b]$. If $\mu(Z) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on Z such that $|f(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is Z–subordinate to δ.*

Proof. For each positive integer n, let $Z_n = \{x \in Z : n-1 \leq |f(x)| < n\}$ and let $\epsilon > 0$. For each n, choose an open set O_n such that $Z_n \subset O_n$ and $\mu(O_n) < \epsilon/n2^n$. Let $\delta(x) = \rho(x, CO_n)$ for $x \in Z_n$. Suppose that $\mathcal{P}$ is Z–subordinate to δ. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in Z_n and compute

$$|f(\mathcal{P})| \leq \sum_{n=1}^{\infty} |f(\mathcal{P}_n)| < \sum_{n=1}^{\infty} n\mu(O_n) < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

Lemma 2 *Let $\{f_n\}$ be a pointwise bounded sequence of functions defined on $[a,b]$ and let $Z \subset [a,b]$. If $\mu(Z) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on Z such that $|f_n(\mathcal{P})| < \epsilon$ for all n whenever $\mathcal{P}$ is Z–subordinate to δ.*

Proof. Let $M(x) = \sup\{|f_n(x)|\}$ for each $x \in [a,b]$ and apply the previous lemma.

Lemma 3 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ is pointwise bounded on $[a,b]$. Let $Z \subset [a,b]$ such that $\mu(Z) = 0$. If the sequence $\{F_n\}$ is generalized $\mathcal{P}$-Cauchy on Z, then for each $\epsilon > 0$ there exists a positive function δ on Z such that $|F_n(\mathcal{P})| < \epsilon$ for all n whenever $\mathcal{P}$ is Z-subordinate to δ.*

Proof. We will consider the case in which the sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy on Z; the general case follows easily. Let $\epsilon > 0$. By hypothesis, there exist a positive integer N and a positive function δ_1 on Z such that $|F_n(\mathcal{P}) - F_m(\mathcal{P})| < \epsilon$ for all $m, n \geq N$ whenever $\mathcal{P}$ is Z–subordinate to δ_1. By Lemma 2, there exists a positive function $\delta_2 < \delta_1$ on Z such that $|f_n(\mathcal{P})| < \epsilon$ for all n whenever $\mathcal{P}$ is Z–subordinate to δ_2. By the definition of the Henstock integral, there exists a positive function δ on $[a,b]$ such that $\delta < \delta_2$ on Z and $|f_n(\mathcal{P}) - F_n(\mathcal{P})| < \epsilon$ for $1 \leq n \leq N$ whenever $\mathcal{P}$ is subordinate to δ. Now suppose that $\mathcal{P}$ is Z–subordinate to δ. Then for $1 \leq n \leq N$,

$$|F_n(\mathcal{P})| \leq |F_n(\mathcal{P}) - f_n(\mathcal{P})| + |f_n(\mathcal{P})| < \epsilon + \epsilon = 2\epsilon,$$

and for $n > N$,

$$|F_n(\mathcal{P})| \leq |F_n(\mathcal{P}) - F_N(\mathcal{P})| + |F_N(\mathcal{P})| < \epsilon + 2\epsilon = 3\epsilon.$$

This completes the proof.

Lemma 4 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges pointwise to f on $[a,b]$. If $\{F_n\}$ is generalized $\mathcal{P}$-Cauchy on $[a,b]$, then there exists an increasing sequence $\{E_k\}$ of closed sets in $[a,b]$ such that $\mu(Z) = 0$ where $Z = [a,b] - \bigcup_{k=1}^{\infty} E_k$ and for each k, the sequence $\{f_n\}$ converges uniformly to f on E_k and the sequence $\{F_n\}$ is $\mathcal{P}$-Cauchy on E_k.*

Proof. First write $[a,b] = \bigcup_{k=1}^{\infty} A_k$ where each A_k is measurable and $\{F_n\}$ is $\mathcal{P}$–Cauchy on each A_k, then use Egorov's Theorem to write $[a,b] = \bigcup_{j=1}^{\infty} B_j \cup Z_0$ where each B_j is measurable, $\{f_n\}$ converges uniformly to f on each B_j, and $\mu(Z_0) = 0$. By reducing a doubly indexed sequence to a sequence, $[a,b] = \bigcup_{j=1}^{\infty} C_j \cup Z_0$ where $\{f_n\}$ converges uniformly to f on each C_j and $\{F_n\}$ is $\mathcal{P}$–Cauchy on each C_j.

For each j, let $C_j = \bigcup_{i=1}^{\infty} D_j^i \cup Z_j$ where $\{D_j^i\}$ is an increasing sequence of closed sets and $\mu(Z_j) = 0$. Finally, define $E_k = \bigcup_{j=1}^{k} D_j^k$ for each k and $Z = \bigcup_{j=0}^{\infty} Z_j$. On each E_k, the sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy and $\{f_n\}$ converges uniformly to f. This completes the proof since $[a,b] = \bigcup_{k=1}^{\infty} E_k \cup Z$, $\{E_k\}$ is an increasing sequence of closed sets, and $\mu(Z) = 0$.

Theorem 5 *Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a,b]$, let $F_n(x) = \int_a^x f_n$ for each n, and suppose that $\{f_n\}$ converges pointwise to f on $[a,b]$. Then the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ if and only if the sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$.*

Proof. Suppose first that the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$. By the Simple Convergence Theorem, the sequence $\{F_n\}$ converges pointwise on $[a,b]$. For each $x \in [a,b]$, let $M_x = \sup_n\{|f_n(x)|\}$ and for each positive integer k, let $E_k = \{x \in [a,b] : k-1 \le M_x < k\}$. Note that each E_k is a measurable set. Since $[a,b] = \bigcup_{k=1}^{\infty} E_k$, it is sufficient to prove that the sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy on E_k for each k.

Fix k and let $\epsilon > 0$. Choose a positive function δ_1 on $[a,b]$ such that $|f_n(\mathcal{P}) - F_n(\mathcal{P})| < \epsilon$ for all n whenever $\mathcal{P}$ is subordinate to δ_1. By Egorov's Theorem, there exists an open set O_k such that $\{f_n\}$ converges uniformly on $E_k - O_k$ and $\mu(O_k) < \epsilon/k$. Choose a positive integer N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \ge N$ and all $x \in E_k - O_k$ and define

$$\delta(x) = \begin{cases} \delta_1(x), & \text{if } x \in E_k - O_k; \\ \min\{\delta_1(x), \rho(x, CO_k)\}, & \text{if } x \in E_k \cap O_k. \end{cases}$$

Suppose that $\mathcal{P}$ is E_k–subordinate to δ and that $m, n \ge N$. Let $\mathcal{P}_1$ be the subset of $\mathcal{P}$ that has tags in $E_k - O_k$ and let $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$. Now use Henstock's Lemma to compute

$$\begin{aligned} |F_n(\mathcal{P}) - F_m(\mathcal{P})| &\le |F_n(\mathcal{P}) - f_n(\mathcal{P})| + |f_n(\mathcal{P}) - f_m(\mathcal{P})| \\ &\qquad + |f_m(\mathcal{P}) - F_m(\mathcal{P})| \\ &\le \epsilon + |f_n(\mathcal{P}_1) - f_m(\mathcal{P}_1)| + |f_n(\mathcal{P}_2) - f_m(\mathcal{P}_2)| + \epsilon \\ &< \epsilon + \epsilon\,(b-a) + 2k\mu(O_k) + \epsilon \\ &< \epsilon\,(b-a+4). \end{aligned}$$

Therefore the sequence $\{F_n\}$ is $\mathcal{P}$–Cauchy on E_k.

Now suppose that the sequence $\{F_n\}$ is generalized $\mathcal{P}$–Cauchy on $[a,b]$. Choose an increasing sequence $\{E_k\}$ of closed sets as in Lemma 4 and let

$Z = [a,b] - \bigcup_{k=1}^{\infty} E_k$. Let $\epsilon > 0$ and let $\epsilon_k = \epsilon\, 2^{-k-1}/(b-a+2)$ for each positive integer k. For each k, choose a positive integer m_k and a positive function δ'_k on E_k such that $|f_m(x) - f_n(x)| < \epsilon_k$ for all $m, n \geq m_k$ whenever $x \in E_k$ and $|F_n(\mathcal{P}) - F_m(\mathcal{P})| < \epsilon_k$ for all $m, n \geq m_k$ whenever $\mathcal{P}$ is E_k-subordinate to δ'_k. Without loss of generality, we may assume that the sequence $\{m_k\}$ is increasing. For each k, there exists a positive function δ_k on $[a,b]$ such that $\delta_k < \delta'_k$ on E_k and $|f_n(\mathcal{P}) - F_n(\mathcal{P})| < \epsilon_k$ for $n = 1, 2, \cdots, m_k$ whenever $\mathcal{P}$ is subordinate to δ_k. By Lemmas 2 and 3, there exists a positive function δ_Z on Z such that $|f_n(\mathcal{P})| < \epsilon/4$ and $|F_n(\mathcal{P})| < \epsilon/4$ for all n whenever $\mathcal{P}$ is Z-subordinate to δ_Z. Let $E_0 = \emptyset$ and define a positive function δ on $[a,b]$ by

$$\delta(x) = \begin{cases} \delta_k(x), & \text{if } x \in E_k - E_{k-1}; \\ \delta_Z(x), & \text{if } x \in Z. \end{cases}$$

Suppose that $\mathcal{P}$ is subordinate to δ and fix n. Define

$$\mathcal{P}_Z = \{(x,I) \in \mathcal{P} : x \in Z\} \quad \text{and} \quad \mathcal{P}_k = \{(x,I) \in \mathcal{P} : x \in E_k - E_{k-1}\}$$

for each k. We then have

$$\begin{aligned} |f_n(\mathcal{P}) - F_n(\mathcal{P})| &\leq \textstyle\sum_{k=1}^{\infty} |f_n(\mathcal{P}_k) - F_n(\mathcal{P}_k)| + |f_n(\mathcal{P}_Z)| + |F_n(\mathcal{P}_Z)| \\ &\leq \textstyle\sum_{k=1}^{\infty} |f_n(\mathcal{P}_k) - F_n(\mathcal{P}_k)| + \epsilon/2. \end{aligned}$$

Now fix k. If $n \leq m_k$, then $|f_n(\mathcal{P}_k) - F_n(\mathcal{P}_k)| < \epsilon_k$ by the choice of δ_k; and if $n > m_k$, then

$$\begin{aligned} |f_n(\mathcal{P}_k) - F_n(\mathcal{P}_k)| &\leq |f_n(\mathcal{P}_k) - f_{m_k}(\mathcal{P}_k)| + |f_{m_k}(\mathcal{P}_k) - F_{m_k}(\mathcal{P}_k)| \\ &\quad + |F_{m_k}(\mathcal{P}_k) - F_n(\mathcal{P}_k)| \\ &< \epsilon_k\,(b-a) + \epsilon_k + \epsilon_k = \epsilon\, 2^{-k-1}. \end{aligned}$$

It follows that

$$|f_n(\mathcal{P}) - F_n(\mathcal{P})| \leq \sum_{k=1}^{\infty} |f_n(\mathcal{P}_k) - F_n(\mathcal{P}_k)| + \epsilon/2 < \sum_{k=1}^{\infty} \epsilon\, 2^{-k-1} + \epsilon/2 = \epsilon.$$

Hence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$.

It should be pointed out that the status of the hypothesis $\{F_n\}$ is equi ACG_* on $[a,b]$ is not known. That is, it has not been shown that this hypothesis is equivalent to $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ nor has a counterexample been produced.

References

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