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On the Descriptive Definition of the Burkill Approximately Continuous Integral

Bullen [1] gave various equivalent definitions of the Burkill approximately continuous integral, which we shall denote by P^*_{ap}. In this paper, we will point out that the D^*_{ap} integral or a descriptive definition of P^*_{ap} defined in [1] is really not equivalent to the P^*_{ap} integral, but more restricted than the latter. If we replace $[ACG^*_{ap}]$ as in [1] by ACG^*_{ap} defined as in [4] (Definition 22.6), we will get another version of the D^*_{ap} integral. Let it be denoted by D^{**}_{ap}. D^{**}_{ap} is more restricted than D^*_{ap} because ACG^*_{ap} is more restricted than $[ACG^*_{ap}]$.

All of this nonequivalence is caused by the very definition of AC^*_{ap} in [1], and that in [4]; the latter will be denoted by AC^{**}_{ap}. The adequate definition of AC^*_{ap} is essential. The author is working on a paper on this topic, and the recent works [2] and [5] have contributed to the theory. We shall assume that the reader is familiar with the relevant definitions involving the integral in [1] and [4].

1. Prerequisites

For definitions of the P^*_{ap}-integral, the R^*_{ap}-integral and their equivalence see [1].

The next 2 definitions are repeated without change from [1] while the 3rd definition is taken from [4].

Definition 1 *Let $F : [a,b] \to \mathbb{R}$ be given.*

(a) Let E be a closed subset of $[a,b]$. Then $F \in AC^*_{ap}(E)$, closed, if and only if (i) $F \in AC(E)$, (ii) for all λ, $0 < \lambda < 1$, there exists, on each closed contiguous interval of E, $[a_n, b_n]$, a set E^λ_n and an $M^\lambda > 0$, $|E^\lambda_n| > (1-\lambda)(b_n - a_n)$, such that for all $x_n \in E^\lambda_n$,

$$\sum_{n\in\mathbb{N}} |F(x_n) - F(a_n)| < M^\lambda, \quad \textit{and} \quad \sum_{n\in\mathbb{N}} |F(b_n) - F(x_n)| < M^\lambda.$$

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(b) $F \in [ACG^*_{ap}]$ on $[a,b]$ if and only if there exist closed sets $E_n, n = 1, 2, \ldots$ such that $[a,b] = \cup_{n\in\mathbb{N}} E_n$, and $F \in AC^*_{ap}(E_n)$, $n \in \mathbb{N}$.

Definition 2 *If $f : [a,b] \to \mathbb{R}$, then $f \in D^*_{ap}$, f is D^*_{ap}-integrable, if and only if there exists $F \in C_{ap}([a,b])$, $F \in [ACG^*_{ap}]$ and $F'_{ap} = f$ almost everywhere; then*

$$\int_a^x f = F(x) - F(a).$$

Here $C_{ap}([a,b])$ denotes the family of all approximately continuous functions on $[a,b]$.

Definition 3 *Let X be closed in $[a,b]$. A function $F : [a,b] \to \mathbb{R}$ is said to be $AC^{**}_{ap}(X)$ if and only if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for all $\alpha_1 < \beta_1 \le \alpha_2 < \ldots < \beta_p$, points of X, if $\sum_{k=1}^{p}(\beta_k - \alpha_k) < \eta$, then for every $\lambda \in (0,1)$ there exist measureable $E^\lambda_k \subset [\alpha_k, \beta_k]$ with $\alpha_k, \beta_k \in E^\lambda_k$ and $|E^\lambda_k| > (1-\lambda)(\beta_k - \alpha_k)$ for $1 \le k \le p$ and satisfying*

$$\sum_{k=1}^{p} \omega(F; E^\lambda_k) < \varepsilon,$$

where $\omega(F; E^\lambda_k) = \sup\{|F(x) - F(y)|;\ x, y \in E^\lambda_k\}$. A function F is said to be $ACG^{**}_{ap}([a,b])$ if and only if $[a,b] = \cup_{i=1}^{\infty} X_i$ where each X_i is closed and F is $AC^{**}_{ap}(X_i)$ for each i.See [4], page 139, Definition 22.6.)

Correspondingly, we define the D^{**}_{ap} integral.

In Theorem 4.5 of [1], page 245 it is asserted that AC^*_{ap} and AC^{**}_{ap}. But Theorem 4.5 is not correct because the δ chosen is not independent of λ. Actually AC^{**}_{ap} is stronger than AC^*_{ap}. We will prove later that this condition together with C_{ap} is no less than AC^*.

In Theorem 4.10 of [1] it is asserted that D^*_{ap} is equivalent to P^*_{ap}. But the proof of Theorem 4.10 is not valid because in the theorem of Tolstoff [7], the portion Q of a perfect set P depends on ε, but what we need in the definition of $[ACG^*_{ap}]$ is that Q must be independent of ε. (See [7] page 657.) We will prove in the following that $D^{**}_{ap} \subset D^*_{ap} \subset P^*_{ap}$ and both inclusions are proper.

2. The nonequivalence of D^{**}_{ap}, D^*_{ap} and P^*_{ap}

For the definition of $AC^*(X)$ see [4].

Proposition 1 *If $F \in C_{ap}([a,b])$ and $AC^{**}_{ap}(X)$ where X is closed in $[a,b]$, then $F \in AC^*(X)$.*

Proof. According to 3, for every $\varepsilon > 0$, there exists $\eta > 0$, such that for $\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \ldots \le \alpha_p < \beta_p$, points of X, if $\sum_{k=1}^{p}(\beta_k - \alpha_k) < \eta$, then for $\lambda = 1/2^n$ where $n = 1, 2, \ldots$ there exists $E_k^{1/2^n}$ such that $E_k^{1/2^n} \subset [\alpha_k, \beta_k]$, $\alpha_k, \beta_k \in E_k^{1/2^n}$, $|E_k^{1/2^n}| > (1 - 2^{-n})(\beta_k - \alpha_k)$, $k = 1, 2, \ldots, p$ and satisfying

$$\sum_{k=1}^{p} \omega(F; E_k^{1/2^n}) < \varepsilon.$$

Put $E_k = \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} E_k^{1/2^n}$, then we have

$$|E_k| = \beta_k - \alpha_k.$$

Hence there exists $E_k \subset [\alpha_k, \beta_k]$, $\alpha_k, \beta_k \in E_k$, and $|E_k| = \beta_k - \alpha_k$, $k = 1, 2, \ldots, p$ such that

$$\sum_{k=1}^{p} \omega(F; E_k) \le \varepsilon.$$

It follows that $\sum_{k=1}^{p} \omega(F; [\alpha_k, \beta_k]) \le 2\varepsilon$. Otherwise, there exist $y_1, y_2 \in [\alpha_K, \beta_K]$ for some $K \in \{1, 2, \ldots, p\}$ and such that $|F(y_1) - F(y_2)| > 2\varepsilon - \sum_{k \ne K} \omega(F; E_k)$. But since F is approximately continuous at y_1, y_2, there exist $x_i \in D_{y_i} \cap E_K$ with D_{y_i} having density 1 at y_i, $i = 1, 2$, such that $|F(x_i) - F(y_i)| < \varepsilon/2$. Hence

$$|F(x_1) - F(x_2)| \ge |F(y_1) - F(y_2)| - \varepsilon > 2\varepsilon - \sum_{k \ne K} \omega(F; E_k) - \varepsilon = \varepsilon - \sum_{k \ne K} \omega(F; E_k).$$

That means $\sum_{k=1}^{p} \omega(F; E_k) > \varepsilon$ which is a contradition.

Proposition 2 *$D_{ap}^{**} \subset D_{ap}^{*}$ and the inclusion is proper.*

Proof. The inclusion is because of Proposition 1, while the properness of the inclusion will be proved by the following Example 2.

Proposition 3 *$D_{ap}^{*} \subset P_{ap}^{*}$ and the inclusion is proper.*

Proof. The inclusion is proved in [1]. We prove the properness by giving in Example 1 a function satisfying P_{ap}^{*} but not D_{ap}^{*}.

Example 1 We denote Cantor's ternary set on $[a, b]$ by P, and we describe associated intervals as follows.

Step 1. Let I_1 be the middle open third of $[a, b]$; let O_1 be the center of I_1; let J_{11}, J_{12} be the other closed thirds of $[a, b]$ at the left and right of I_1 respectively.

Step 2. Let I_{11} be the middle open third of J_{11} with centre O_{11}; let J_{111}, J_{112} be the other thirds of J_{11} at the left and right of I_{11} respectively, and likewise we get $I_{12}, O_{12}, J_{121}, J_{122}$.

Continuing this procedure, in general, after n similar steps, we have got $J_{1\alpha_1\alpha_2\dots\alpha_n}$, let $I_{1\alpha_1\alpha_2\dots\alpha_n}$ be the middle open third of $J_{1\alpha_1\alpha_2\dots\alpha_n}$ with centre $O_{1\alpha_1\alpha_2\dots\alpha_n}$; and $J_{1\alpha_1\alpha_2\dots\alpha_n\alpha_{n+1}}$ be the other thirds of $J_{1\alpha_1\alpha_2\dots\alpha_n}$ at the left or right of $I_{1\alpha_1\alpha_2\dots\alpha_n}$ according to whether α_{n+1} is 1 or 2.

Finally let $K(\alpha_1, \alpha_2, \dots, \alpha_n)$ be the number of "α_i's" in $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ equal to 2. Define

$$F(x) = \begin{cases} 1 & x \in O_1 \\ \frac{1}{n+1} & x \in O_{1\alpha_1\alpha_2\dots\alpha_n}; \alpha_i = 1,2,\ i = 1,2,\dots,n;\ n \in \mathbb{N}; \\ 0 & x \in [a,b] \setminus \bigcup (L_{\alpha_1\alpha_2\dots\alpha_n}, R_{\alpha_1\alpha_2\dots\alpha_n});\ \text{where} \\ & L_{\alpha_1\alpha_2\dots\alpha_n} = O_{1\alpha_1\alpha_2\dots\alpha_n} - (1/2^{K(\alpha_1,\alpha_2,\dots,\alpha_n)+1})|I_{1\alpha_1\alpha_2\dots\alpha_n}|, \\ & R_{\alpha_1\alpha_2\dots\alpha_n} = O_{1\alpha_1\alpha_2\dots\alpha_n} + (1/2^{K(\alpha_1,\alpha_2,\dots,\alpha_n)+1})|I_{1\alpha_1\alpha_2\dots\alpha_n}|, \\ & \alpha_i = 1,2 \text{ for } i = 1,2,\dots,n;\ n \in \mathbb{N}. \end{cases}$$

Extend F to $[a,b]$ by requiring it to be linear on $[L_{\alpha_1\alpha_2\dots\alpha_n}, O_{1\alpha_1\alpha_2\dots\alpha_n}]$ and on $[O_{1\alpha_1\alpha_2\dots\alpha_n}, R_{\alpha_1\alpha_2\dots\alpha_n}]$.

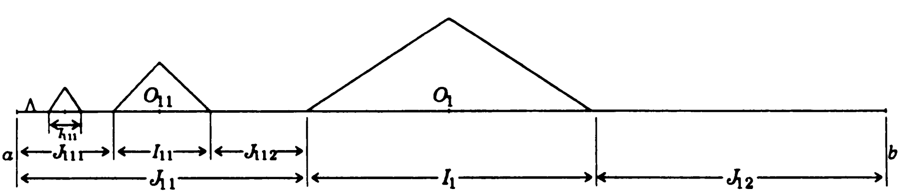
For example $F(O_{12122112}) = 1/8$; F linear on

$$[O_{12122112} - (1/2^5.3^8), O_{12122112}] \text{ and } [O_{12122112}, O_{12122112} + (1/2^5 \cdot 3^8)];$$

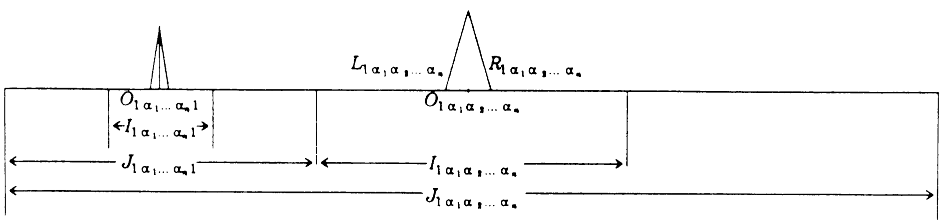
$F(x) = 0$ when x is the points of $I_{12122112}$ other than the above-mentioned points.

Or we illustrate $F(x)$ as follows.

On $I_1, I_{11}, I_{111}, \dots, I_{11\dots1}, \dots$, the graph of F consists of triangles with bases of length $|I_1|, |I_{11}|, \dots, |I_{11\dots1}|$ and heights $1, 1/2, 1/3, \dots, 1/n$ respectively,



On each of the intervals $I_{1\alpha_1\dots\alpha_n}, I_{1\alpha_1\dots\alpha_n1}, I_{1\alpha_1\dots\alpha_n11}, \dots, I_{1\alpha_1\dots\alpha_n11\dots1}, \dots$ that are extracted from $J_{1\alpha_1\alpha_2\dots\alpha_n}$ as the above intervals are from $[a,b]$, the graph of F consists of a triangle with that interval as base and height $1/(n+k)$, where k is the number of 1's following the sequence $1\alpha_1\dots\alpha_n$ that defines the base.



Now let us prove the following.

(1) $F'_{ap}(x) = f(x)$ nearly everywhere, $f \in P^*_{ap}$ and $F(x) = P^*_{ap} - \int_a^x f(y)dy$.

(2) F is not $[ACG^*_{ap}]([a, b])$, so $F'_{ap}(x)$ cannot be D^*_{ap}-integrable on $[a, b]$.

Proof of (1) $F(a) = 0$. For every $x \in I_{1\alpha_1\alpha_2\ldots\alpha_n}$ except $O_{1\alpha_1\alpha_2\ldots\alpha_n}$, $L_{\alpha_1\alpha_2\ldots\alpha_n}$ and $R_{\alpha_1\alpha_2\ldots\alpha_n}$, $F'_{ap}(x)$ exists, since $F(x)$ is linear there, and $F \in C_{ap}(I_{1\alpha_1\ldots\alpha_n})$. For every $x \in P$, except the endpoints of any $I_{1\alpha_1\ldots\alpha_n}$, there exists a sequence $J_{1\alpha_1}, J_{1\alpha_1\alpha_2}, \ldots, J_{1\alpha_1\alpha_2\ldots\alpha_n}, \ldots$ including x as their interior point, and we will prove $F'_{ap}(x)$ exists and equals 0.

Lemma 1 *Let $x \in P$ and suppose x is not the end point of any $I_{1\alpha_1\ldots\alpha_n}$. Then*

$$\lim_{n\to\alpha} K(\alpha_1, \alpha_2, \ldots, \alpha_n) = \infty \text{ and } \lim_{n\to\infty} [n - K(\alpha_1, \alpha_2, \ldots, \alpha_n)] = \infty,$$

where for each n, x is an interior point of $J_{1\alpha_1\alpha_2\ldots\alpha_n}$.

Proof. If $\lim_{n\to\infty} K(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq \infty$, then there is p such that $\alpha_p = 2$ and $\alpha_k = 1$ for $k > p$. Then x is the right end point of $I_{1\alpha_1\alpha_2\ldots\alpha_{p-1}}$, giving a contradiction. Likewise for $\lim_{n\to\infty}[n - K(\alpha_1\alpha_2\ldots\alpha_n)]$. □

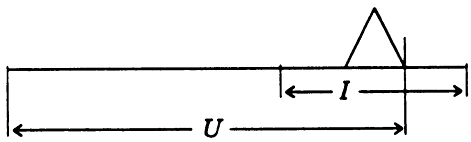
For every above mentioned x, we will show $F'_{ap}(x) = 0$ by taking for our set D_x of density 1 at x the set $\{y : F(y) = 0\}$. To show that $\{y : F(y) = 0\}$ does indeed have density 1 at x, let $J_{1\alpha_1\alpha_2\ldots\alpha_n}$ be as in Lemma 1 such that $K(\alpha_1, \alpha_2, \ldots, \alpha_n) > M + 1$ for any given $M \in \mathbb{N}$. Then for every neighbourhood U of x with $U \subset J_{1\alpha_1\alpha_2\ldots\alpha_n}$, U only includes points belonging to P or to intervals $I_{1\alpha_1\alpha_2\ldots\alpha_n\alpha_{n+1}\ldots\alpha_{n+p}}$ with

$$K(\alpha_1, \alpha_2, \ldots, \alpha_n) > M + 1.$$

When $y \in U \cap P$, we have $F(y) = 0$. When y belongs to the intersection of U and any $I_{1\alpha_1\ldots\alpha_n\alpha_{n+1}\ldots\alpha_{n+p}} \subset J_{1\alpha_1\ldots\alpha_n}$, let I denote $I_{1\alpha_1\alpha_2\ldots\alpha_n\alpha_{n+1}\ldots\alpha_{n+p}}$, and K denote $K(\alpha_1, \alpha_2, \ldots, \alpha_n) > M + 1$; then

$$|\{y : y \in I, F(y) \neq 0\}| = 2^{-K(\alpha_1,\ldots,\alpha_n,\alpha_{n+1},\ldots,\alpha_{n+p})}|I| \leq 2^{-K}|I|.$$

Now, let us estimate $|\{y : y \in I \cap U, F(y) = 0\}|/|I \cap U|$. It reaches its minimum when U just includes $\{y : y \in I, F(y) \neq 0\}$ and half of $\{y : y \in I, F(y) = 0\}$, as shown below



In this case,

$$\frac{|\{y : y \in I \cap U, F(y) = 0\}|}{|I \cap U|} =$$

$$2^{-1}|I|(1 - \frac{2^{-K(\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1},\ldots,\alpha_{n+p})})}{2^{-1}|I|(1 + 2^{-K(\alpha_1,\ldots,\alpha_n,\alpha_{n+1},\ldots,\alpha_{n+p})})}$$

$$\geq (1 - 2^{-K})/(1 + 2^{-K}) = (1 + 2^{-K} - 2^{-K+1})/(1 + 2^{-K}) =$$

$$1 - (2^{-K+1}/1 + 2^{-K}) > 1 - 2^{-K+1} > 1 - 2^{-M}$$

. Hence, whenever $I \cap U \neq \phi$, in every case, we have $|\{y : y \in I \cap U, F(y) = 0\}|/|I \cap U| > 1 - 2^{-M}$.

Now

$$U = (U \cap P) \cup \{\cup(I \cap U)\},$$

the union in the second parentheses being over all above mentioned intervals I, these intervals being nonoverlapping.

Hence

$$|\{y : y \in U, F(y) = 0\}|/|U| =$$

$$(|U \cap P| + \sum |\{y : y \in I \cap U, F(y) = 0\}|)/(|U \cap P| + \sum |I \cap U|) \geq$$

$$\sum |\{y : y \in I \cap U, F(y) = 0\}|/\sum |I \cap U| > 1 - 2^{-M}$$

and that means D_x has density 1 at x and hence $F'_{ap}(x) = 0$.

Since $\lim_{n\to\infty} \max\{F(x); x \in I_{1\alpha_1\alpha_2\ldots\alpha_n}\} = 0$ for any sequence $\{I_{1\alpha_1\alpha_2\ldots\alpha_n}\}$, and $F(x) = 0$ in P, F is C_{ap} on $[a, b]$, and even continuous on $[a, b]$.

Summing up the above, we have $F \in C_{ap}([a, b])$, $F'_{ap}(x) = f(x)$ exists n.e. (except a countable set) on $[a, b]$, so $F \in M_f^{\#}$ as well as $M_{\#,f}$ and hence

$$F(y) = P^*_{ap} - \int_a^y f(x)dx.$$

Proof of (2) (i.e. F is not $[ACG^*_{ap}]([a,b])$, so $F'_{ap}(x)$ cannot be D^*_{ap}-integrable on $[a,b]$.

Lemma 2 *$F \in [ACG^*_{ap}]$ on $[a,b]$, if and only if for every perfect R there exists a portion Q of R such that $F \in AC^*_{ap}(Q)$.*

The proof of the Lemma is word for word the same as that involving ACG^* in Saks [6]. □

Suppose, to obtain a contradition, that $F \in [ACG^*_{ap}]$ on $[a,b]$. By Lemma 2, there is a portion Q of P with $F \in AC^*_{ap}(Q)$. If (c,d) is the smallest interval including Q, then there exists a $J_{1\overline{\alpha}_1\overline{\alpha}_2\ldots\overline{\alpha}_n} \subset (c,d)$ and $(c,d)\backslash Q$ will include all $I_{1\overline{\alpha}_1\ldots\overline{\alpha}_n 11\ldots 1}$ with every suffix following $\overline{\alpha}_n$ being "1".

Given any $I_{1\overline{\alpha}_1\ldots\overline{\alpha}_n 11\ldots 1}$, denotedby (ℓ,r) and any $\lambda < 2^{-K(\overline{\alpha}_1,\ldots,\overline{\alpha}_n)-1}$, suppose $E^{\lambda}_{(\ell,r)} \subset [\ell,r]$ is taken to be as in the definition of AC^*_{ap}. Now

$$|E^{\lambda}_{(\ell,r)}| > (1-\lambda)|(\ell,r)|,$$

and the length of the base of the triangle that graphs $F(x)$ in (ℓ,r) is equal to $2^{-K(\overline{\alpha}_1,\ldots,\overline{\alpha}_n)}|(\ell,r)| > 2\lambda|(\ell,r)|$; hence there exists y belonging to $E^{\lambda}_{(\ell,r)}$ as well as to the central half of the base of the above triangle. Hence $F(y) > 2^{-1}(n+p+1)^{-1}$, where p is the number of "1's" following $\overline{\alpha}_n$, and so we have

$$|F(\ell) - F(y)| > 2^{-1}(n+p+1)^{-1}.$$

And so

$$\sum |F(\ell) - F(y)| > \sum_{p=1}^{\infty} 2^{-1}(n+p+1)^{-1} = \infty,$$

the sum being for all $I_{1\overline{\alpha}_1\ldots\overline{\alpha}_n 1\ldots 1}$. Hence F is not $AC^*_{ap}(Q)$, giving a contradiction, i.e. F is not $[ACG^*_{ap}]$, and $F'_{ap}(x) = f(x)$ is not D^*_{ap}-integrable, completing Example 1, and proving Proposition 3. □

Note If we replace $\frac{1}{n+1}$ by $1/\big(n - K(\alpha_1,\alpha_2,\ldots,\alpha_n)+1\big)$ when $x = O_{1\alpha_1\ldots\alpha_n}$; then F also satisfies P^*_{ap} but not D^*_{ap}, and at the left end points of $I_{1\alpha_1\alpha_2\ldots\alpha_n}$, F is only approximately continuous but not continuous.

Proof of Proposition 2 By giving the following Example 2.

Example 2 Let P, $I_{1\alpha_1\alpha_2\ldots\alpha_n}$ and $O_{1\alpha_1\alpha_2\ldots\alpha_n}$ be given as in Example 1, and define

$$F(x) = \begin{cases} 1 & x \in O_1 \text{ or } x \in O_{1\alpha_1\alpha_2\ldots\alpha_n}; \alpha_i = 1,2,\ i = 1,2,\ldots,n,\ n \in \mathbb{N} \\ 0 & x \in [a,b] \setminus \cup \Big(O_{1\alpha_1\ldots\alpha_n} - \frac{|I_{1\alpha_1\ldots\alpha_n}|}{2^{n+1}}, O_{1\alpha_1\ldots\alpha_n} + \frac{|I_{1\alpha_1\ldots\alpha_n}|}{2^{n+1}}\Big) \end{cases}$$

Extending F to $[a,b]$ by requiring it to be linear on the intervals contiguous to the set consisting of all above mentioned points.

It is easy to prove that $F'(x)$ is D^*_{ap} integrable but not D^{**}_{ap} integrable. First, the structure of F shows that it is C_{ap} on $[a,b]$, and then F is AC^*_{ap} on every closure of $I_{1\alpha_1\alpha_2\ldots\alpha_n}$. Lastly, for proving F to be AC^*_{ap} on P (the Cantor ternary set in $[a,b]$), for every $\lambda \in (0,1)$, there exists $N \in \mathbb{N}$ such that $(1/2^N) < \lambda$. For every interval $I_{1\alpha_1\alpha_2\ldots\alpha_n}$ contiguous to P let us denote it by $[\ell, r]$, and denote

$$E^{\lambda}_{[\ell,r]} = \{x; x \in [\ell, r], F(x) = 0\}.$$

We have $|E^{\lambda}_{[\ell,r]}| > (1-\lambda)|[\ell,r]|$ for every $I_{1\alpha_1\alpha_2\ldots\alpha_n}$ with $n \geq N$ and the number of those $I_{1\alpha_1\alpha_2\ldots\alpha_n}$ with $n < N$ is finite. Hence

$$\sum \omega(F; E^{\lambda}_{[\ell,r]}) < \infty$$

and F is AC^*_{ap} on P. So we have proven the D^*_{ap} integrability of $F'(x)$. On the other hand, by the theorem corresponding to our Lemma 2 in §9, Chapter VII of Saks [6], we know that F is not ACG^* on $[a,b]$. Hence $F'(x)$ is not D^{**}_{ap}-integrable. □

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