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A Note on Darboux Functions

In a recent paper [2], B. Kirchheim and T. Natkaniec showed that if Martin's axiom holds, then there is a Darboux function f such that $f+g$ is not Darboux if g is a nowhere constant, continuous function. What they proved can be reformulated as follows. If $\mathcal{G}$ is a family of nowhere constant, continuous functions, then there is a Darboux function f such that $f+g$ is not Darboux ($g \in \mathcal{G}$) as long as $|\mathcal{G}|$ does not exceed the size of the least partition of $\mathbb{R}$ into nowhere dense subsets. A well-known corollary of Martin's axiom is that this latter cardinal is 2^ω. In this note we prove the result under the condition that $|\mathcal{G}|$ is not large in another sense, namely, there is at least one cardinal between $|\mathcal{G}|$ and the continuum.

Theorem 1 *If $\mathcal{G}$ is a family of nowhere constant, continuous functions with $|\mathcal{G}|^+ < 2^\omega$ then there exists a Darboux function f such that $f+g$ is not Darboux whenever $g \in \mathcal{G}$.*

Notation. We use the standard axiomatic set theory notation. Cardinals are identified with initial ordinals, 2^ω is the cardinal of the continuum. κ^+ is the cardinal successor of κ.

Lemma 1 *If V is a vector space over $\mathbb{Q}$, $|V| = \lambda > \kappa^+$, $\mathcal{F}$ is a family of $V \to V$ functions, $|\mathcal{F}| = \kappa$, then there exists a set $X \subseteq V$ of size λ, such that no κ translates of $\{f(x): x \in X,\ f \in \mathcal{F}\}$ cover V.*

Proof. Let $\mu < \lambda$ be either κ^+ or κ^{++} such that $cf(\lambda) \neq \mu$ hold. Let W be a subspace of V of dimension λ and co-dimension μ. V/W can be written as the increasing union of subspaces of size $< \mu$, $V/W = \cup\{V_\alpha/W: \alpha < \mu\}$. For $x \in W$, the set $\{f(x): f \in \mathcal{F}\}$ is of size at most κ, so it is contained in one of the V_α (as $cf(\mu) > \kappa$). Put $x \in W_\alpha$ if $\{f(x): f \in \mathcal{F}\} \subseteq V_\alpha$. This gives an increasing decomposition $W = \cup\{W_\alpha: \alpha < \mu\}$. Frome the Claim below it follows that some

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W_α is of size λ , and as κ many translates of V_α/W cannot cover V/W, we are done. **Claim.** *If* $|W| = \lambda$, $\mu < \lambda$, $\mu \neq cf(\lambda)$, *W is the increasing union of the sets* $\{W_\alpha : \alpha < \mu\}$, *then* $|W_\alpha| = \lambda$ *for some* $\alpha < \mu$.

Proof. Obvious, if λ is regular or at least $\mu < cf(\lambda)$. Assume that λ is singular, and $\mu > cf(\lambda)$. Let $\{\lambda_\xi : \xi < cf(\lambda)\}$ be a sequence of cardinals converging to λ. For each $\alpha < \mu$, let $\xi(\alpha)$ be minimal such that $|W_\alpha| \leq \lambda_{\xi(\alpha)}$. As $\mu > cf(\lambda)$, for a cofinal set of $\alpha < \mu$, $\xi(\alpha) = \xi$. But then, as the sequence W_α is increasing, $|W_\alpha| \leq \lambda_\xi$ for *all* $\alpha < \mu$, so $|W| \leq \mu\lambda_\xi < \lambda$.

Lemma 2 *If* $\{g_\alpha : \alpha < \kappa\}$ *is a family of nowhere constant, continuous functions,* $\kappa^+ < 2^\omega$, *then there are real numbers* c_α *such that if I is an interval, d is a real number, then for continuum many* $x \in I$, *there is no* $\alpha < \kappa$ *such that* $g_\alpha(x) + c_\alpha = d$.

Proof. Enumerate the rational intervals as $\{I_n : n < \omega\}$. Construct the functions $h_\alpha(x, n)$ in such a way that $h_\alpha(x, n) \in I_n$, $g_\alpha(h_\alpha(x, n)) \neq g_\alpha(x)$, and $h_\alpha(x, n) \neq h_\beta(y, n)$ unless $\alpha = \beta$ and $x = y$. This is possible, as by hypothesis, $g_\alpha(x)$ misses every value in every interval 2^ω times, so a straightforward diagonalization of length 2^ω works.

Let $\mathcal{F}$ be the family of functions which can be written in the form $g_\alpha(H_1(x)) - g_\beta(H_2(x))$ where $H_1(x)$, $H_2(x)$ are composed from the functions $h_\gamma(x, n)$ ($\gamma < \kappa$, $n < \omega$). Here, as usual, $x \mapsto f_1(\ldots f_n(x)\ldots)$ is called the composition of the functions $f_1, \ldots, f_n$. Clearly, $|\mathcal{F}| \leq \kappa$. We can, therefore, apply 1, and get an appropriate set X. Let $Z = \{H(x) : x \in X\}$ where H runs through the finite compositions of the h_γ's. By the statement of 1, we can select, by transfinite induction on $\alpha < \kappa$, reals c_α such that

$$g_\alpha(h_\alpha(x, n)) + c_\alpha \neq g_\beta(x) + c_\beta, \qquad (\beta < \alpha, x \in Z). \tag{1}$$

To finish the proof, we must show that if I_n, d are given, there are 2^ω elements x of I_n such that $g_\alpha(x) + c_\alpha \neq d$ for $\alpha < \kappa$. The set $Y = Z \cap I_n$ is a subset of I_n of size 2^ω. If $y = y_0 \in Y$, and $g_{\alpha_0}(y_0, n) + c_{\alpha_0} = d$ for some $\alpha_0 < \kappa$, define $y_1 = h_{\alpha_0}(y_0, n)$. If $g_{\alpha_1}(y_1) + c_{\alpha_1} = d$ for some $\alpha_1 < \kappa$, then $\alpha_1 \neq \alpha_0$ by the choice of h_{α_0}, and $\alpha_0 < \alpha_1$ is also impossible by (1). So, $\alpha_1 < \alpha_0$. Continuing, we get real numbers $y_0, y_1, \ldots$, and a decreasing sequence of ordinals $\alpha_0, \alpha_1, \ldots$. As there is no infinite decreasing sequence of ordinals, we eventually find an element z such that $g_\alpha(z) + c_\alpha = d$ holds for no $\alpha < \kappa$.

As the functions h_α were supposed to get different values at different arguments, the only possibility for getting the same value $z \in I_n$ as above from two different y's is that one of them occurs in the chain obtained from the other. By the injectivity of the functions h_α the chains are disjoint and countable, so, as each of them must contain an appropriate $z \in I_n$, there are 2^ω of them.

Proof of 1 If $\mathcal{G} = \{g_\alpha : \alpha < \kappa\}$ is a family of nowhere constant, continuous functions, let $\{c_\alpha : \alpha < \kappa\}$ be selected according to 2. If we enumerate the pairs of reals and natural numbers as $\{(r_\alpha, n_\alpha) : \alpha < 2^\omega\}$, and select by transfinite induction on $\alpha < 2^\omega$ an s_α such that $s_\alpha \in I_{n_\alpha}$, $s_\alpha \neq s_\beta$ $(\beta < \alpha)$, and that for no $\gamma < \kappa$ does $g_\gamma(s_\alpha) + c_\gamma = r_\alpha$ hold, then by defining $f(s_\alpha) = r_\alpha$, f will take every value on every interval. On the places x, where f is undefined, let $f(x)$ be any value different from $g_\alpha(x) + c_\alpha$ $(\alpha < \kappa)$. Clearly, the range of $f - g_\alpha$ will be everywhere dense, but will exclude c_α.

With the method applied here one can prove other translation results, like the following one.

Theorem 2 *One can assign a real number $c(g)$ to every continuous, nowhere linear function g such that the union of the graphs of the functions $g(x) + c(g)$ does not contain a straight segment.*

Proof. Enumerate those functions as $\{g_\alpha : \alpha < 2^\omega\}$, the rational intervals as $\{I_n : n < \omega\}$, and the real numbers as $\{r_\alpha : \alpha < 2^\omega\}$.

By transfinite induction on $\alpha < 2^\omega$ we select $c_\alpha \in \mathbb{R}$, $b(\alpha, n) \in I_n$, and $h_{\beta,\alpha}(x_1, x_2, n) \in I_n \cap \mathbb{Q}$ for $x_1 \neq x_2 \in \mathbb{Q}$, $n < \omega$, $\beta \leq \alpha$. Assume that all these objects have been selected for the ordinals smaller than α.

Select c_α so that

$$c_\alpha + g_\alpha(r_\beta) \neq b(\beta, n) \qquad (\beta < \alpha, n < \omega) \tag{2}$$

and

$$g_\alpha(z) + c_\alpha \neq y_1 + (z - x_1)\frac{y_2 - y_1}{x_2 - x_1} \tag{3}$$

for $\beta_1, \beta_2 < \alpha$, $x_1 \neq x_2 \in \mathbb{Q}$, $n < \omega$, where $y_i = g_{\beta_i}(x_i) + c_{\beta_i}$, $z = h_{\beta_1,\beta_2}(x_1, x_2, n)$.

This selection is possible, as (2–3) exclude only $< 2^\omega$ values of c_α.

Next, select $b(\alpha, n) \in I_n$ so that

$$b(\alpha, n) \neq c_\beta + g_\beta(r_\alpha) \qquad (\beta \leq \alpha, n < \omega). \tag{4}$$

This is, again, possible, for the same reason.

Finally, let $z = h_{\beta,\alpha}(x_1, x_2, n) \in I_n \cap \mathbb{Q}$ be such that $z \neq x_1, x_2$ and $g_\beta(z) + c_\beta$, $g_\alpha(z) + c_\alpha$ are not on the segment determined by $(x_1, g_\beta(x_1) + c_\beta)$ and $(x_2, g_\alpha(x_2) + c_\alpha)$. This can be done, as our functions are nowhere linear, continuous.

We claim that the union of the graphs of the translated functions $g_\alpha(x) + c_\alpha$ does not contain a straight segment.

Assume that we are given the vertical segment $\{r\} \times I$. If $I = I_n$ and $r = r_\alpha$, the point $(r_\alpha, b(\alpha, n))$ is missed by (2) and (4).

If the segment is nonvertical, let $\alpha_1 \leq \alpha_2$ be the first two ordinals, such that for some different rational $x_1 \neq x_2$, the points $(x_1, g_{\alpha_1}(x_1) + c_{\alpha_1}), (x_2, g_\alpha(x_2) + c_{\alpha_2})$ are on the segment, let I_n be the projection of the segment on the x axis. Put $y_1 = g_{\alpha_1}(x_1) + c_{\alpha_1}$, $y_2 = g_{\alpha_2}(x_2) + c_{\alpha_2}$,

$$z = h_{\alpha_1,\alpha_2}(x_1, x_2, n), \qquad u = y_1 + (z - x_1)\frac{y_2 - y_1}{x_2 - x_1}.$$

We claim that (z, u) is an uncovered point on the segment. Notice that $z \in I_n \cap \mathbb{Q}$. If $g_\alpha(z) + c_\alpha = u$ for some α, then $\alpha > \alpha_2$ is impossible by (3). $\alpha = \alpha_1$ or α_2 is impossible by the choice of z, and $\alpha_1 < \alpha < \alpha_2$ or $\alpha < \alpha_1$ contradict the minimality of α_2, α_1, resp.

But this is probably old stuff.

References

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