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On Some Topologies of O'Malley's Type on the Plane

Let $(X, \mathcal{T})$ be a topological space. A real function f defined on X is said to be

- $\mathcal{T}$-quasi-continuous at a point $x_0 \in X$ iff for every $\varepsilon > 0$ and for any neighbourhood $U \in \mathcal{T}$ of the point x_0 there exists $V \in \mathcal{T}$ such that $\emptyset \neq V \subset U$ and $|f(x) - f(x_0)| < \varepsilon$ for every $x \in V$,

- $\mathcal{T}$-cliquish at $x_0 \in X$ iff for every $\varepsilon > 0$ and for any neighbourhood $U \in \mathcal{T}$ of the point x_0 there exists $V \in \mathcal{T}$ such that $\emptyset \neq V \subset U$ and $osc_V f < \varepsilon$.

If $\mathcal{T}$ is the Euclidean topology on $\mathbb{R}^n$, we will write "quasi-continuous", "cliquish" instead of "$\mathcal{T}$-quasi-continuous" and "$\mathcal{T}$-cliquish".

In the present paper we study the families of $\mathcal{T}$-quasi-continuous functions and $\mathcal{T}$-cliquish functions defined on $\mathbb{R}^2$ with some topologies of density type.

I. S. Kempisty proved in [8] that if every x-section of $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f_x(t) = f(x,t)$ and every y-section of f, $f^y(t) = f(t,y)$ is quasi-continuous then f is quasi-continuous, too. Note that the analogous theorem is not true for density topology d (see e.g. [2], p. 20, for definitions).

Example 1 *Under Martin's Axiom there exists a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that*

1. *all f_x and f^y sections of f are d-quasi-continuous,*
2. *f is not $d \times d$-cliquish (thus f is not $d \times d$-quasi-continuous),*
3. *f is not measurable [6]*

The following Lemma is proved in [5], p. 13.

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Lemma 1 *A real function f defined on $\mathbb{R}^n$ $(n = 1, 2)$ is measurable iff for any $\varepsilon > 0$ and for any measurable set $A \subset \mathbb{R}^n$ with positive measure there exists a measurable subset B of A with positive measure for which $osc_B f < \varepsilon$.*

Corollary 1 *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable iff it is d-cliquish [6].*

Neither implication holds in $\mathbb{R}^2$ as is seen from Example 1 and the following

Example 2 *There exists a measurable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is not $d \times d$-cliquish.*

Indeed, let Q denote the set of all rationals and $A = \{(x, y) : \ y = x + s \ \text{ for } \ s \in \mathbb{R} \setminus Q\}$. As is easily seen, both A and its complement are $d \times d$ -dense (this is a consequence of Steinhaus's Theorem [13], cf [1]), whence we may take f – the characteristic function of A.

Let us recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the property (G) iff for every $\varepsilon > 0$ and for any perfect set A with positive measure there exists an open interval J such that $m(A \cap J) > 0$ and $osc_{int_d(A) \cap J} f < \varepsilon$ [5]. Since every Baire 1 function has the property (G), every d-continuous function has this property.

Theorem 1 *Let f be a real function defined on $\mathbb{R}^2$. If each section f_x has the property (G) and each section f^y is d-quasi-continuous, then f is $d \times d$-cliquish.*

Proof. Fix $(x_0, y_0) \in \mathbb{R}^2$ and d-neigbourhoods I of x_0 and J of y_0, $\varepsilon > 0$ and $\delta = \varepsilon/4$. Let A be a perfect subset of J with $m(A) > 0$. Since f_x has the property (G), for any $x \in \mathbb{R}$ there is an open interval $J_x = (p_x, q_x)$ such that $p_x, q_x \in Q$, $m(J_x \cap A) > 0$ and $osc_{J_x \cap int_d(A)} f_x < \delta$. Consequently, there exists an interval K for which the set $B = \{x \in I : \ J_x = K\}$ has positive outer measure. Fix $x \in B \cap \varphi^*(B)$, where $\varphi^*(B)$ denotes the set of all points of outer density of B, and fix $y \in K \cap int_d(A)$. Since f^y is d-quasi-continuous, the set $C = (f^y)^{-1}(f(x, y) - \delta, f(x, y) + \delta)$ is measurable and $m(C \cap B) > 0$. Note that, by the d -quasi-continuity of f^y, the d-neighbourhood $I \cap \varphi^*(B)$ of x contains a d-open set D the sets $D = I \cap C \cap \varphi^*(B)$. Moreover, $E = K \cap A \cap int_d(A)$ is d-open and $D \times E \subset I \times J$. For $(t, u) \in (B \cap D) \times E$ we have

$$|f(t, u) - f(x, y)| \leq |f(t, u) - f(t, y)| + |f(t, y) - f(x, y)| < 2\delta.$$

Since f^y is d-quasi-continuous and B is d-dense in D, $|f(t, u) - f(x, y)| < 2\delta$ for every $(t, u) \in D \times E$ and consequently, $osc_{D \times E}(f) < 4\delta = \varepsilon$.

□

Corollary 2 *If all sections f_x of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are d-continuous and all sections f^y are d-quasi-continuous, then f is $d \times$ d-cliquish.*

Example 3 *There exists a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that each section f_x is d-continuous and each section f^y is d-cliquish but f is not $d \times d$-cliquish [6].*

Example 4 *There exists a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ for which all sections f_x and f^y are d-continuous and which is not $d \times d$-quasi-continuous.*

Indeed, let $g : \mathbb{R} \longrightarrow [0,1]$ be a d-continuous function such that $g(x) = 0$ for $x \in Q$ and $m(g^{-1}(0)) = 0$ (see [16]). Let us put $f(x,y) = g(y-x)$ for $(x,y) \in \mathbb{R}^2$. Evidently, all x and y sections of f are d-continuous. Fix $x \in \mathbb{R}$, $s \in \mathbb{R} \setminus Q$ with $g(s) \neq 0$, a $d \times d$-neighbourhood $I \times J$ of $(x, x+s)$ and $\varepsilon = g(s)/2$. By Steinhaus's Theorem we obtain that $int(J_1 - I_1) \neq \emptyset$ for any non-empty $d \times d$-open subset $I_1 \times J_1$ of $I \times J$. Consequently, there exist $t \in Q$, $v \in J_1$ and $u \in I_1$ such that $v = u + t$. Then we have $f(u,v) = g(v-u) = 0$ and $|f(u,v) - f(x, x+s)| = g(s) > \varepsilon$. Thus f is not $d \times d$-quasi-continuous.

In the first version of this paper (which was written a few years ago) we posed the problem of measurability of $d \times d$ -quasi-continuous functions of two variables. W. Wilczyński has recently solved this problem in the negative.

Example 5 *There exists a non-measurable and $d \times d$-quasi-continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ [15].*

II. O'Malley defined in [10] the d_{xy} topology as the family of all measurable subsets U of $\mathbb{R}^2$ for which all sections $U_x = \{t : (x,t) \in U\}$ and $U^y = \{t : (t,y) \in U\}$ of U are d-open. Of course, every non-empty, d_{xy}-open set has positive measure. On the other hand, for every measurable set $A \in \mathbb{R}^2$ having positive measure there exists an non-empty, d_{xy}-open subset B of A (see [5]). Therefore we have the following

Proposition 1 *A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable iff it is d_{xy}-cliquish.*

Let us recall that the collection of all sets of the form $U \setminus I$, where U is open and I has measure zero forms a topology on $\mathbb{R}$. This topology is called the $*$-topology in the sense of Hashimoto for the σ-ideal of all measure zero sets [7]. Thus we can define two new topologies on $\mathbb{R}^2$ of O'Malley type.

- $d^*_{xy} = \{U \in d_{xy} : U_x \text{ and } U^y \text{ are } * \text{-open for all } x, y \in \mathbb{R}\}$,
- $d^0_{xy} = \{U \in d_{xy} : U_x \text{ and } U^y \text{ are open for all } x, y \in \mathbb{R}\}$,

Note that d^0_{xy} is a proper subclass of d^*_{xy} and d^*_{xy} is a proper subclass of d_{xy}. Additionally, simple examples show that the classes $Cq(d^0_{xy})$, $Cq(d^*_{xy})$ and $Cq(d_{xy})$ of all cliquish functions (with respect to proper topologies) are pairwise distinct.

III. Now we shall construct some category analogue of d_{xy} topology. We say that a set $A \subset \mathbb{R}$ is q-open iff A is of the form $U \setminus I$, where U is open and I is of first category. The class q of all q-open sets is equal to the $*$-topology of Hashimoto with respect to the ideal of all first category sets. Note that every second category set A having the Baire property contains an non-empty, q-open subset $B \subset A$. Moreover, the following theorem is proved in [4].

Lemma 2 *Let f be a real function defined on $\mathbb{R}^n$, $n = 1,2$. Then f has the Baire property iff for every $\varepsilon > 0$ and for each second category set A having the Baire property there exists a second category set $B \subset A$ having the Baire property with $osc_B f < \varepsilon$.*

Thus we have the following

Proposition 2 *A real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the Baire property iff it is q-cliquish.*

Example 6 *There exists a function with the Baire property which is not $q \times q$-cliquish.*

Indeed, let f and A be defined as in Example 2. A is residual and therefore $q \times q$-dense. By Piccard's Theorem [12], $\mathbb{R}^2 \setminus A$ is $q \times q$-dense. Thus f fulfills all requirements.

Proposition 3 *Every $q \times q$-cliquish function has the Baire property.*

Proof. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $q \times q$-cliquish and $A \subset \mathbb{R}^2$ be a second category set with the Baire property. Then A we may assume that A is the difference of an open and non-empty set G and a first category set K. Choose open intervals I, J such that $I \times J \subset G$. Let B be $q \times q$-open, non-empty subset of $I \times J$ with $osc_B f < \varepsilon$. Then $B \setminus K$ is of second category and has the Baire property, $B \setminus K \subset A$ and $osc_{B \setminus K} < \varepsilon$. Consequently, f has the Baire property by Lemma 2.

□

Let q_{xy} be the collection of all subsets U of $\mathbb{R}^2$ with the following properties:

1. U has the Baire property,
2. all sections U_x and U^y of U are q-open.

Lemma 3 *If $A \in q_{xy}$ is of first category, then it is empty.*

Proof. If $A \subset \mathbb{R}^2$ is of first category, then by the Kuratowski-Ulam Theorem (see e.g. [11]), nearly every section of A is of first category, and hence (being

q-open) it is empty. Thus $A \subset A_1 \times A_2$, where A_1, A_2 are linear first category sets. But then every section of A is of first category. Hence every section of A is empty and consequently, A is empty.

□

Theorem 2 *The collection q_{xy} forms a topology on $\mathbb{R}^2$.*

Proof. This theorem can be proved in the similar way as Theorem 1 [10]. However we present here a shorter proof which has been communicated to us by one of the referees. It is clear that only one condition needs to be verified. Namely, for some index set T, if U_t belongs to q_{xy} for each $t \in T$, then $U = \bigcup_{t \in T} U_t$ has the Baire property. We may assume that each U_t is non-empty. Then $U_t = (G_t \setminus A_t) \cup B_t$, where G_t is non-empty and open and A_t, B_t are first category sets. Put

$$G = \bigcup G_t, \quad B = \bigcup B_t, \quad X = \bigcup (G_t \setminus A_t).$$

Now $(\mathbb{R}^2 \setminus \overline{G_t}) \cap U_t$ is again in q_{xy}, and being a set of first category, must be empty by Lemma 3. Thus $B_t \subset \overline{G_t} \setminus G_t$, and hence $(B \setminus G) \subset \bar{G} \setminus G$, that is $B \setminus G$ is a first category set. Let $(S_n)_n$ run through the set of rational discs on the plane. Denote $T_n = \bigcap \{A_t : S_n \subset G_t\}$. Then by $G_t = \bigcup \{S_n : S_n \subset G_t\}$, we have

$$X = \bigcup_{t \in T} \bigcup_{S_n \subset G_t} (S_n \setminus A_t) = \bigcup_{S_n \subset G} (S_n \setminus T_n).$$

Hence X admits the Baire property, moreover

$$G \setminus \bigcup T_n \subset X \subset X \cup (G \cap B) \subset G,$$

where $\bigcup T_n$ is of first category. Therefore $X \cup (G \cap B)$ has the Baire property as well, and finally by

$$U = X \cup (G \cap B) \cup (B \setminus G)$$

we get the statement.

□

Example 7 *The family of all subsets B of $\mathbb{R}^2$ having the Baire property (which are measurable) and such that all sections B_x and B^y of B are d-open (q-open) does not form topology on $\mathbb{R}^2$.*

Indeed, let (A, B) be a partition of $\mathbb{R}$ into two disjoint sets: a F_σ subset of first category A and a G_δ subset of measure zero B. As B is uncountable, there exists a subset C of B without he Baire property (see e.g. [11], p. 24; Theorem 5.5). Let us put $U_c = (A \cup \{c\}) \times \mathbb{R}$ for each $c \in C$. Then U_c are Borel subsets of $\mathbb{R}^2$ with d-open sections but $\bigcup_{c \in C} U_c$ does not have the Baire property. The similar example shows that the family of all measurable subsets of $\mathbb{R}^2$ with q-open sections does not form a topology.

Finally we define two new collections of subsets of $\mathbb{R}^2$

- q_{xy}^+ is the collection of all subsets U of $\mathbb{R}^2$ having the Baire property, for which all sections are open in the $\mathcal{J}$-density topology introduced by Wilczyński (see [14] for definitions and basic properties),
- q_{xy}^0 is the collection of all subsets of $\mathbb{R}^2$ having the Baire property, for which all sections are open in the Euclidean topology.

In the similar way as in Theorem 2 one can prove that q_{xy}^+ forms topology on $\mathbb{R}^2$ (note that the continuous real functions relative to this topology are the separately $\mathcal{I}$-approximately continuous functions). Thus q_{xy}^0 is a topology too and we have the following proper inclusions

$$q_{xy}^0 \subset q_{xy} \subset q_{xy}^+.$$

Proposition 4 *For a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ the following conditions are equivalent:*

(i) f has the Baire property,

(ii) f is q_{xy}-cliquish,

(iii) f is q_{xy}^+-cliquish.

Proof. This is an immediate consequence of Lemma 2 and the following (easy to see) fact: every second category set having the Baire property contains a non-empty, q_{xy}-open subset (cf [3], Theorem 1).

□

Finally note that every q_{xy}^0-cliquish function is q_{xy} -cliquish but the characteristic function of the set $Q \times Q$ is in the class $Cq(q_{xy}) \setminus Cq(q_{xy}^0)$.

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References

[1] R. Anantharaman and J.P. Lee, *Planar sets whose complements do not contain a dense set of lines*, Real Anal. Exchange 11, No 1 (1985–86), 168–179.

[2] A.M. Bruckner, *Differentiation of Real Functions*, Springer–Verlag, Berlin 1978.

[3] Z. Grande, *Sur la propriété de Baire des fonctions de deux variables*, Bull. Ac. Pol. Math. 25, No 4 (1977), 349–354.

[4] Z. Grande, *Sur la propriété de Baire d'une fonction de deux variables ponctuellement discontinue par rapport à une variable*, Bull. Ac. Pol. Math. 25, No 6 (1977), 533–537.

[5] Z. Grande, *La mesurabilité des fonctions de deux variable et de la superposition* $F(x, f(x))$, Dissert. Math. 159 (1978), 1–50.

[6] Z. Grande, *Sur les fonctions approximativement quasi-continues*, Revue Roum. Math. Pures et Appl., 34(1989), 17–22.

[7] H. Hashimoto, *On the * topology and its applications*, Fund. Math. 91, No. 1 (1976), 275–284.

[8] S. Kempisty, *Sur les fonctions semi-continues par rapport à chacune de deux variables*, Fund. Math. 14 (1929), 237–241.

[9] K. Kuratowski, *Topologie*, Vol. I. Warszawa, PWN 1958.

[10] R.J. O'Malley, *Separate approximate continuity and strong approximate continuity*, Colloquium Mat. 50, No. 1 (1985), 129–132.

[11] J.C. Oxtoby, *Measure and Category*, New York–Heidelberg–Berlin, 1971.

[12] S. Piccard, *Sur les ensembles de distance*, Memoires Neuchâtel Université 1938–1939.

[13] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. 1 (1920), 99–104.

[14] W. Wilczyński, *A category analogue of the density topology, approximate continuity and the approximate derivative*, Real Anal. Exchange 10, No. 2 (1984–85), 241–266.

[15] W. Wilczyński, *A non-measurable* $d \times d$ *quasi-continuous function*, Bull. Soc. Sci. Lett. Łódź, 40.9, 1990, 147–150.

[16] Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. 69 (1950), 1–54.