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On the Sums and Products of Darboux Baire*1 Functions

Let **R** denote the set of all real numbers. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be Baire* 1 ([5]) if for every perfect set $A \subset \mathbf{R}$ there is an open interval I such that $A \cap I \neq \emptyset$ and the restricted function $f/(A \cap I)$ is continuous. Obviously, the sum and the product of two Baire* 1 functions are Baire* 1 functions.

Let us settle some of the notation to be used in the article.

B_1^* - the class of all Baire* 1 functions,

D - the class of all Darboux functions,

C - the class of all continuous functions,

$DB_1^* + DB_1^* = \{f + g;\ f, g \in DB_1^*\}$, $DB_1^* \cdot DB_1^* = \{fg;\ f, g \in DB_1^*\}$,

$M(DB_1^*) = \{f;$ for every $g \in DB_1^*$ the sum $f + g \in DB_1^*\}$,

$P(DB_1^*) = \{f;$ for every $g \in DB_1^*$ the product $fg \in DB_1^*\}$,

$E(DB_1^*) = \{f \in B_1^*; f$ has a zero in each subinterval in which it changes sign$\}$ ([2]),

$F(DB_1^*) = \{f \in DB_1^*;$ if f is discontinuous from the right (resp. left) at $x = a$ then $f(a) = 0$ and there is a sequence $x_n \searrow a$ ($y_n \nearrow a$) such that $f(x_n) = 0$ ($f(y_n) = 0$)$\}$ ([3]).

Since the constant functions $f = 0$ and $g = 1$ are in DB_1^*, $M(DB_1^*) \cup P(DB_1^*) \subset DB_1^*$.

In this paper we characterize the families $DB_1^* + DB_1^*$, $DB_1^* \cdot DB_1^*$, $M(DB_1^*)$, and $P(DB_1^*)$. Moreover, we prove that every function $f \in DB_1^*$ is quasicontinuous, i.e. for every $x \in \mathbf{R}$, for every $r > 0$, and for every neighborhood U of x there is a nonempty open set $V \subset U$ such that $f(V) \subset (f(x) - r, f(x) + r)$ ([4]).

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Theorem 1 *Suppose that a Darboux function $f : \mathbf{R} \to \mathbf{R}$ is such that for every open interval I the preimage $f^{-1}(I)$ is F_σ as well as G_δ. Then f is a quasicontinuous function.*

Proof. Fix $x \in \mathbf{R}$, an open neighborhood U of x and $r > 0$. If $I = (f(x) - r, f(x) + r)$ then $f^{-1}(I)$ is an F_σ and G_δ set. Since f has the Darboux property, the set $f^{-1}(I)$ is bilaterally c-dense-in-itself. So, there is an interval $(a, b) \subset U \cap f^{-1}(I)$ ([6]), and consequently, $(a, b) \subset U$ and $f((a, b)) \subset I$. This completes the proof.

Corollary 2 *Every Darboux, Baire* 1 function is quasicontinuous.*

Proof. It suffices to remark that every Baire* 1 function f is such that $f^{-1}(I)$ is an F_σ- and G_δ set for every open interval I ([7]).

Remark 1 *Rosen's theorem in [8] follows immediately from Theorem 1.*

Theorem 3 *The equality $DB_1^* + DB_1^* = B_1^*$ is true.*

Proof. Fix $f \in B_1^*$. Since f is a Baire* 1 function, the interior of the set $C(f)$ of all continuity points of f is dense. We may assume that $C(f) \neq \mathbf{R}$. Consequently, the set $D(f) = \mathbf{R} - C(f)$ is nowhere dense. In every open component (a, b) of the interior $\operatorname{int} C(f)$ of the set $C(f)$ with $a, b \in \mathbf{R}$ there are two sequences $a_n \searrow a$ and $b_n \nearrow b$ such that $a_1 < b_1$. Analogously, in every component (a, b) of the set $\operatorname{int} C(f)$ with $a = -\infty$ or $b = \infty$ there is a sequence $b_n \nearrow b$ or respectively $a_n \searrow a$. If $a, b \in \mathbf{R}$ then there is a continuous function $f_{ab} : (a, b) \to \mathbf{R}$ such that:

$$f_{ab}(x) = 0 \text{ for } x \in (a_1, b_1) \text{ or } x = a_i \text{ or } x = b_i, \;\; i = 1, 2, \ldots ; \tag{1}$$

$$(f + f_{ab})([a_{n+1}, a_n]) \supset [-n, n], \;\; n = 1, 2, \ldots ; \tag{2}$$

$$(f + f_{ab})([b_n, b_{n+1}]) \supset [-n, n], \;\; n = 1, 2, \ldots ; \tag{3}$$

If $a = -\infty$ ($b = \infty$), then we define such f_{ab} which satisfies only the conditions (1), (3) ((1), (2)). Let us put

$$g(x) = \begin{cases} f(x) + f_{ab}(x) & \text{in the component } \;\; (a, b) \text{ of } \operatorname{int} C(f) \\ f(x) & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} -f_{ab}(x) & \text{in the component } \;\; (a, b) \text{ of } \operatorname{int} C(f) \\ 0 & \text{otherwise} \end{cases} .$$

Evidently, $f = g + h$ and the functions g, h are continuous at each point $x \in \text{int } C(f)$. So, for every perfect set A with $A \cap \text{int } C(f) \neq \emptyset$ there is an open interval $I \subset \text{int } C(f)$ such that $I \cap A \neq \emptyset$ and $g/(A \cap I)$ and $h/(A \cap I)$ are continuous. If A is a perfect set contained in the closure cl $D(f)$ of the set $D(f)$ then $g/A = f/A$ and $h/A = 0$. Since $f \in B_1^*$, there is an open interval I such that $A \cap I \neq \emptyset$ and $g/(A \cap I) = f/(A \cap I)$ is continuous. Obviously, $h/(A \cap I) = 0$ is also continuous. So, g, h are Baire* 1 functions. From (2), (3) it follows that the right cluster sets

$$C^+(g, x) = \{y \in \bar{\mathbf{R}};\ \text{there is a sequence } x_n \searrow x \text{ with } g(x_n) \to y\},$$

and the left cluster sets $C^-(g, x) = \{y \in \bar{\mathbf{R}};$ there is a sequence $x_n \nearrow x$ with $g(x_n) \to y\}$ are equal to $\bar{\mathbf{R}}$ for $x \in$ cl $D(f)$. By (1) $0 \in C^+(h, x) \cap C^-(h, x)$ for $x \in$ cl $D(f)$. Since the functions g, h are continuous at every point $x \in \mathbf{R}\text{cl } D(f) = \text{int } C(f)$, the functions g, h have the Darboux property ([1], pp. 8-9, Thm. 1.1.).

Theorem 4 *The following equality $M(DB_1^*) = C$ is true.*

Proof. The proof is the same as the proof of Bruckner's theorem 3.2 in [1] on p. 14.

Theorem 5 *The following equality $DB_1^* \cdot DB_1^* = E(B_1^*)$ is true.*

Proof. The proof is the same as the proof of Ceder's theorem in [2]. It is necessary to remark that the functions g, h in Ceder's proof in [2] are Baire* 1 whenever $f \in E(B_1^*)$.

Theorem 6 *The following equality $P(DB_1^*) = F(B_1^*)$ is true.*

Proof. The proof is a modification of the proof of Fleissner's theorem in [3]. If $f \in F(B_1^*)$ and $g \in DB_1^*$ then $fg \in D$ ([3]). Since $gf \in B_1^*$, the sufficiency is proven. For the proof of the necessity we consider two cases.

Case 1. Suppose that $f \in P(DB_1^*)$ is discontinuous from the right at a point a and $f(x) > 0$ on $(a, a+r]$ $(r > 0)$. There is $K > 0$ such that there is a sequence $p_n \searrow a$ with $f(p_n) \to K \neq f(a)$. Set

$$g(x) = \begin{cases} 1/f(a+r) & \text{for } x \geq a + r \\ 1/f(x) & \text{for } x \in (a, a+r) \\ 1/K & \text{for } x \leq a \end{cases}$$

Then $g \in DB_1^*$, but $f(a)g(a) \neq 1$ and $f(x)g(x) = 1$ on $(a, a+r)$. So $fg \notin D$.

Case 2. Suppose that f is discontinuous from the right at a with $f(a) > 0$, and there is a sequence $p_n \searrow a$ with $f(p_n) = 0$. Let $E = \{x : x > a,\ f(x) < f(a)/2\}$. Since $f \in DB_1^*$, there are ([7]) disjoint closed intervals $I_n = [a_n, b_n]$, $n = 1, 2, \dots$, contained in E and such that $a < a_{n+1} < b_{n+1} < a_n$, $n = 1, 2, \dots, a_n \searrow a$, $b_n \searrow a$. Consequently, there is a function $g \in DB_1^*$ such that $0 < g(x) \leq 1$ for $x \in \bigcup_n I_n$, $g(x) = 0$ for $x \in (a, \infty) - \bigcup_n I_n$, and $g(x) = 1$ for $x \leq a$. Then $g(a)f(a) = f(a)$ and $f(x)g(x) < f(a)/2$ for $x > 0$. So $fg \notin D$. It suffices to consider only the two cases, since we can suppose that $f \geq 0$ (in the contrary case we can consider f^2).

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