INROADS

Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 225–231

T. Šalát, Department of Algebra and Number Theory, Faculty of Mathematics and Physics, Comenius University, Mlynská dolina 842 15 Bratislava, Czechoslovakia

J. Tóth and L. Zsilinszky, Department of Mathematics, College of Education, Farská 19, 949 74 Nitra, Czechoslovakia

Metric Space of Metrics Defined on a Given Set

1. Introduction

Let X be a non-void set. Denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all metrics on X. We can introduce a metric d^* on $\mathcal{M}$ as follows:

If $d, d' \in \mathcal{M}$ then

$$d^*(d,d') = \min\{1, \sup_{x,y\in X} |d(x,y) - d'(x,y)|\}.$$

First of all recall some basic definitions and notations.

The symbol $t_\alpha (\alpha > 0)$ stands for a trivial metric on X i.e. $t_\alpha(x,x) = 0$ for every $x \in X$ and $t_\alpha(x,y) = \alpha$ for $x \neq y, x,y \in X$. Furthermore if $d \in \mathcal{M}$ and $\varepsilon > 0$, denote by $K(d,\varepsilon) = \{d' \in \mathcal{M} : d^*(d,d') < \varepsilon\}$ (a ball in $\mathcal{M}$) and $\overline{K}(d,\varepsilon) = \{d' \in \mathcal{M} : d^*(d,d') \leq \varepsilon\}$ (a closed ball in $\mathcal{M}$).

Denote by $|B|$ the cardinality of the set B and by $\mathcal{P}(B)$ the power set of B.

If $|X| = 1$, then obviously $|\mathcal{M}(X)| = 1$. Therefore in the following we shall always assume that $|X| \geq 2$.

Denote by $\aleph_0$ and c the cardinality of the set of all positive integers $\mathbb{N}$ and the set of all real numbers $\mathbb{R}$, respectively.

If $\mathcal{M}_1 \subset \mathcal{M}$, then $\mathcal{M}_1$ is considered as a metric space with the metric $d^*|_{\mathcal{M}_1 \times \mathcal{M}_1}$ (a metric subspace of $\mathcal{M}$).

2. Lemmas

In our further considerations some lemmas will take an important place. It is easy to check, that the space $(\mathcal{M}, d^*)$ is not complete. For example, the sequence $\{t_{\frac{1}{k}}\}_{k=1}^{\infty}$ of elements of $\mathcal{M}$ is fundamental, nevertheless has no limit in $\mathcal{M}$. In connection with this we mention some subspaces of $\mathcal{M}$, which are already complete. Suppose $\alpha > 0$ and put

$$\mathcal{H}_\alpha = \{d \in \mathcal{M} : \underset{\substack{x \neq y \\ x,y \in X}}{\forall} d(x,y) \geq \alpha\}$$

Lemma 1 *The subspace $\mathcal{H}_\alpha$ of $\mathcal{M}$ is a complete metric space.*

Proof. Let $\{d_n\}_{n=1}^{\infty}$ be a fundamental sequence of elements of $\mathcal{H}_\alpha$. Then by the definition of the metric d^* for every $x, y \in X$ we have $|d_m(x,y) - d_n(x,y)| \to 0$ $(n, m \to \infty)$. Therefore there exists $d(x,y) = \lim_{n\to\infty} d_n(x,y)$. This function d is a metric on X, since d_n $(n \in \mathbb{N})$ are metrics and $d(x,y) = \lim_{n\to\infty} d_n(x,y) \geq \alpha > 0$ for $x \neq y$. It is now obvious that $d \in \mathcal{H}_\alpha$ and $d^*(d_n, d) \to 0$ as $n \to \infty$.

Remark 1 *It can be seen similarly to the proof of Lemma 1 that the sets $\mathcal{H}_\alpha$ $(\alpha > 0)$ are closed in $\mathcal{M}$.* □

Notice that $\mathcal{H}_{\alpha'} \subset \mathcal{H}_\alpha$, if $0 < \alpha < \alpha'$ thus putting $\mathcal{H} = \bigcup_{\alpha>0} \mathcal{H}_\alpha$ we get $\mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}_{\frac{1}{k}}$. According to Remark 1 the set $\mathcal{H}$ is an F_σ-set in $\mathcal{M}$. This observation will be strengthened in the following lemma.

Lemma 2 *The set $\mathcal{H}$ is an open and dense subset of $\mathcal{M}$.*

Proof. Suppose that $d \in \mathcal{H}$. We prove that d is an inner point of $\mathcal{H}$. Since $\mathcal{H} = \bigcup_{k=1}^{\infty} \mathcal{H}_{\frac{1}{k}}$ there exists an integer $m \geq 1$ such that $d \in \mathcal{H}_{\frac{1}{m}}$. We prove that $K(d, \frac{1}{2m}) \subset \mathcal{H}$. If $d' \in K(d, \frac{1}{2m})$, then for every x, y we have $|d'(x,y) - d(x,y)| < \frac{1}{2m}$. Hence for $x \neq y$ we obtain $d'(x,y) > d(x,y) - \frac{1}{2m} \geq \frac{1}{m} - \frac{1}{2m} = \frac{1}{2m} > 0$, thus $d' \in \mathcal{H}_{\frac{1}{2m}} \subset \mathcal{H}$.

The density of $\mathcal{H}$ can be shown as follows. Let $d \in \mathcal{M}$ and $0 < \varepsilon < 1$. It is sufficient to prove that the ball $K(d, \varepsilon)$ contains an element from $\mathcal{H}$. Assume that $0 < \alpha < \varepsilon$ an choose $\varrho_\alpha = d + t_\alpha$. Then obviously $\varrho_\alpha \in \mathcal{H}$ and $d^*(d, \varrho_\alpha) = \min\{1, \sup\{\alpha\}\} = \alpha < \varepsilon$, so $\mathcal{H} \cap K(d, \varepsilon) \neq \emptyset$. □

Remark 2 *Evidently $\mathcal{M} = \mathcal{H} \cup (\mathcal{M} \setminus \mathcal{H})$ and according to Lemma 2 the set $\mathcal{M} \setminus \mathcal{H}$ is closed and nowhere dense in $\mathcal{M}$. Hence the "substantial part" of the space $\mathcal{M}$ is the set $\mathcal{H}$.*

3. Main Results

We derive the main results which describe the basic properties of the metric space $(\mathcal{M}, d^*)$. First, the cardinality of the space $(\mathcal{M}, d^*)$ will be investigated.

Theorem 1 *(i) If X is a finite set, $|X| \geq 2$, then $|\mathcal{M}(X)| = c$.*

(ii) If X is an infinite set, then $|\mathcal{M}(X)| = 2^{|X|}$.

Proof. (i) Let $|X| = n$ be a finite cardinal ($n \geq 2$), then $c \leq |\mathcal{M}(X)|$, since c is the cardinality of the set of all trivial metrics t_α ($\alpha > 0$) on X. On the other hand $\mathcal{M}(X) \subset {}^{X\times X}\mathbb{R}$, so $|\mathcal{M}(X)| \leq c^{n \cdot n} = c$, thus $|\mathcal{M}(X)| = c$.

(ii) Suppose that X is an infinite set and put $|X| = \alpha$. For every $A \subset X$ define the function $\varrho_A : X \times X \to \mathbb{R}$ as follows. If $x \neq y$, then

$$\varrho_A(x,y) = \varrho_A(y,x) = \begin{cases} 1, & \text{for } x,y \in A \\ 2, & \text{for } x,y \in X \setminus A \\ \Theta, & \text{for } x \in A, y \in X \setminus A, \text{ where } 1 < \Theta < 2 \end{cases}$$

and naturally $\varrho_A(x,x) = 0$ for all $x \in X$. It is easy to verify that $\varrho_A \in \mathcal{M}$, so

$$|\mathcal{M}(X)| \geq \mathcal{P}(X) = 2^{|X|} \tag{1}$$

as $\varrho_A \neq \varrho_{A'}$ for $A \neq A'$, $A, A' \subset X$.

Conversely we have $\mathcal{M}(X) \subset {}^{X\times X}\mathbb{R}$ so

$$|\mathcal{M}(X)| \leq c^{\alpha \cdot \alpha} = c^\alpha = \left(2^{\aleph_0}\right)^\alpha = 2^{\aleph_0 \alpha} = 2^\alpha = 2^{|X|} \tag{1'}$$

From (1),(1′) we get by the Cantor-Bernstein theorem that $|\mathcal{M}(X)| = 2^{|X|}$. □

Theorem 2 *The space $(\mathcal{M}, d^*)$ is dense in itself, moreover each point $d \in \mathcal{M}$ is a point of condensation.*

Proof. Let $d \in \mathcal{M}$ and $0 < \varepsilon < 1$. For $0 < a < \varepsilon$, define $\varrho_a(x,y) = d(x,y) + a$, if $x \neq y$, $x,y \in X$ and $\varrho_a(x,x) = 0$ ($x \in X$). Then obviously $\varrho_a \in \mathcal{M}$ and $d^*(d, \varrho_a) = a < \varepsilon$. Thus $\varrho_a \in K(d,\varepsilon)$ for every $a \in (0,\varepsilon)$. □

Let us mention that a topological (metric) space X is said to be a Baire space, if every non-empty open subset is of the second category in X (see [1]). As we had already said, the space $(\mathcal{M}, d^*)$ is not complete, so there remains a question whether $\mathcal{M}$ is a set of the 2nd category in $(\mathcal{M}, d^*)$. The answer to this question follows.

Theorem 3 *The metric space $(\mathcal{M}, d^*)$ is a Baire space.*

Proof. Let $U \neq \emptyset$ be an open set in $\mathcal{M}$. We shall show that U is a set of the second category in $\mathcal{M}$.

According to Lemma 2 we have $U \cap \mathcal{H} \neq \emptyset$ and so there is an $\alpha_0 > 0$ such that $U \cap \mathcal{H}_{\alpha_0} \neq \emptyset$. Notice that if $\alpha_1 < \alpha_0$ then $\text{int}\mathcal{H}_{\alpha_1} \supset \mathcal{H}_{\alpha_0}$ and consequently $U_0 = U \cap \text{int}\mathcal{H}_{\alpha_1} \neq \emptyset$. Choose $d \in U_0$. Then there is a $\delta > 0$ such that $K(d,\delta) = \{d' \in \mathcal{M}; d^*(d,d') < \delta\} \subset U_0$. Hence the ball $K(d,\delta)$ (in $\mathcal{M}$) is an open subset of $\mathcal{H}_{\alpha_1}$ and according to Lemma 1 it is a set of the 2nd category in $\mathcal{H}_{\alpha_1}$. From this it can be easily deduced that $K(d,\delta)$ is of the 2nd category also in $\mathcal{M}$.

Now it suffices to observe that $U \supset K(d,\delta)$. □

There is a natural question, whether the space $(\mathcal{M}, d^*)$ is separable. The answer depends on the cardinality of the set X, as it is proved in the following theorem.

Theorem 4 *The metric space $(\mathcal{M}, d^*)$ is separable if and only if the set X is finite.*

Proof. If X is finite, let $X = \{x_1, x_2, \ldots, x_n\}$ $(n \geq 2)$ and $p = \binom{n}{2}$. For $\mathbf{u} = (u_1, u_2, \ldots, u_p) \in \mathbb{R}^p$, $\mathbf{v} = (v_1, v_2, \ldots, v_p) \in \mathbb{R}^p$ put

$$\varrho(\mathbf{u}, \mathbf{v}) = \min\{1, \max_{i=1,2,\ldots,p} |u_i - v_i|\} \tag{2}$$

The function $\varrho : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ defined above is a metric equivalent with the Euclidean metric on $\mathbb{R}^p$. Therefore the metric space $(\mathbb{R}^p, \varrho)$ has the same topological properties as the metric space $\mathbb{R}^p$ with the Euclidean metric. Define the mapping $f : (\mathcal{M}, d^*) \to (\mathbb{R}^p, \varrho)$ as follows:

$$f(d) = (d(x_1, x_2), d(x_1, x_3), \ldots, d(x_i, x_j), \ldots, d(x_{n-1}, x_n)) \tag{3}$$

where $d \in \mathcal{M}$ and $1 \leq i < j \leq n$. Since the set X is finite, then

$$d^*(d, d') = \varrho(f(d), f(d')) \tag{4}$$

for every $d, d' \in \mathcal{M}$. According to (4) the function f is an isometric mapping of $\mathcal{M}$ onto the range $H(f)$ of f, and $H(f)$ as a subspace of $(\mathbb{R}^p, \varrho)$ is a separable space. The separability of $\mathcal{M}$ follows now from the fact that $\mathcal{M}$ is a continuous image (under the mapping f^{-1}) of the separable space $H(f)$.

Conversely, let $|X| \geq \aleph_0$. Choose an infinite subset X_0 of X with $X_0 = \{x_1, x_2, \ldots, x_n, \ldots\}$. Arrange all ordered pairs $[x_i, x_j]$ with $i < j$, $i, j \in \mathbb{N}$ into one-to-one sequence $\{P_k\}_{k=1}^{\infty}$. Denote by S the set of all sequences $s = \{s_k\}_{k=1}^{\infty}$ of $\frac{1}{2}$'s and 1's. Then, as is well-known, $|S| = c$.

For $s \in S$ let ϱ_s be a real function defined on $X \times X$ as follows: $\varrho_s(x,x) = 0$ for each $x \in X$, $\varrho_s(P_k) = \varrho_s(x_i, x_j) = \varrho_s(x_j, x_i) = s_k$ if $P_k = [x_i, x_j]$ $(i < j)$. Further $\varrho_s(x,y) = \frac{1}{2}$ provided that x or y belongs to $X \setminus X_0$.

It is easy to check that $\varrho_s \in \mathcal{M}(X)$. Denote by $\mathcal{M}_0 = \mathcal{M}_0(X)$ the set of all ϱ_s $(s \in S)$. Since $\varrho_s \neq \varrho_{s'}$ for $s \neq s'$ we see that $|\mathcal{M}_0| = c$ and $d^*(\varrho_s, \varrho_{s'}) = \frac{1}{2}$ for $s \neq s'$. Hence $\mathcal{M}_0$ is a subset of the power of the continuum of $\mathcal{M}$ consisting of isolated points. The non-separability of $\mathcal{M}$ follows. □

Theorem 5 *The space $(\mathcal{M}, d^*)$ is connected if and only if the set X is finite.*

Proof. Let X be a finite set, $X = \{x_1, x_2, \ldots, x_n\}$ $(n \geq 2)$ and put $p = \binom{n}{2}$. Let f be the mapping from (3). It is easy to check that the range $H(f)$ of the function f is a convex subset of the space $(\mathbb{R}^p, \varrho)$ (see (3)) and therefore is connected. Then the space $(\mathcal{M}, \varrho^*)$ is connected as well, since f^{-1} is continuous (according to (4)) and evidently $\mathcal{M} = f^{-1}(H(f))$.

Let X be an infinite set. Denote by $\mathcal{A}$ the set of all unbounded metrics on X. Obviously $\mathcal{A} \neq \mathcal{M}$ (because $t_1 \in \mathcal{M} \setminus \mathcal{A}$). We prove that $\mathcal{A} \neq \emptyset$.

Let $x_n \in X$ $(n = 1, 2, \ldots)$, $x_i \neq x_j$ $(i \neq j)$ and denote by X_0 the set $\{x_1, x_2, \ldots, x_n, \ldots\}$. Define the mapping $d : X \times X \to \mathbb{R}$ as follows:

$$\begin{aligned}
&d(x_n, x_m) = |n - m|, \ \text{for } n, m = 1, 2, \ldots \\
&d(x, x_n) = d(x_n, x) = n, \ \text{for } x \notin X_0 \\
&d(x, y) = d(y, x) = 1, \ \text{for } x, y \notin X_0, \ x \neq y \\
&d(x, x) = 0, \ \text{for } x \in X
\end{aligned}$$

It can be easily verified that $d \in \mathcal{A}$ so $\mathcal{A} \neq \emptyset$.

Suppose now that $d \in \mathcal{A}$ and $d' \notin \mathcal{A}$. Then d' is a bounded metric on X so necessarily $d^*(d, d') = 1$, thus $K(d, 1) \subset \mathcal{A}$. This implies that the set $\mathcal{A}$ is open in $\mathcal{M}$.

Furthermore if $d_n \in \mathcal{A}$ $(n \in \mathbb{N})$ and $d^*(d_n, d) \to 0$ $(n \to \infty)$ then $d \in \mathcal{A}$. Indeed, supposing $d \notin \mathcal{A}$ we obtain that $d^*(d_n, d) = 1$ $(n \in \mathbb{N})$, which contradicts the convergence of the sequence $\{d_n\}_{n=1}^{\infty}$. Thus the set $\mathcal{A}$ is simultaneously open and closed in $\mathcal{M}$. Further, $\emptyset \neq \mathcal{A} \neq \mathcal{M}$. This means that the space $(\mathcal{M}, d^*)$ is not connected. □

We know, that the space $(\mathcal{M}, d^*)$ is not complete, so is neither compact. For this reason it is a natural question whether the space $(\mathcal{M}, d^*)$ is locally compact.

Theorem 6 *The space $(\mathcal{M}, d^*)$ is locally compact if and only if the set X is finite.*

Proof. Let X be a finite set, $X = \{x_1, x_2, \dots, x_n\}$ $(n \geq 2)$ and put $p = \binom{n}{2}$. We prove a stronger statement, namely that the space $(\mathcal{M}, d^*)$ has the property (z), i.e. $\mathcal{M} = \bigcup_{k=1}^{\infty} G_k$, where G_k are open sets in $\mathcal{M}$ such that $\overline{G_k}$ are compact and $\overline{G_k} \subset G_{k+1}$ $(k \in \mathbb{N})$ (see [2] p.158 Theorem 8.1).

Let f be the function from (3). Denote by

$$G'_m = \{(u_1, u_2, \dots u_p) \in H(f) : \frac{1}{m+1} < u_j < m,\ j = 1, 2, \dots, p\}$$

and $G_m = f^{-1}(G'_m)$ $(m \in \mathbb{N})$, where $H(f)$ is the range of f. The set G'_m is open in $H(f)$, so from the continuity of f (see (2),(4)) it follows that the set G_m is open in $\mathcal{M}$ $(m \in \mathbb{N})$.

We have

$$\overline{G'_m} = \{(u_1, u_2, \dots u_p) \in H(f) : \frac{1}{m+1} \leq u_j \leq m,\ j = 1, 2, \dots, p\}$$

Then the set $\overline{G'_m}$ is a closed and bounded subset of $\mathbb{R}^p$ and therefore it is compact. Since the mapping f is isometric (see (4)) then for every positive integer m we have $\overline{G_m} = f^{-1}(\overline{G'_m})$. Thus $\overline{G_m}$ is a compact subset of $\mathcal{M}$. Furthermore obviously $\overline{G_m} \subset G_{m+1}$ $(m \in \mathbb{N})$ and $\mathcal{M} = \bigcup_{m=1}^{\infty} G_m$.

Suppose now that the set X is infinite and $x_n, y_n \in X$ $(n \in \mathbb{N})$ be one-to-one sequences such that $x_i \neq y_j$ $(i, j = 1, 2, \dots)$

Let $0 < \varepsilon < 1$. For every positive integer k define the mapping $d_k^{(\varepsilon)}$ as follows :

$$\begin{aligned}
&d_k^{(\varepsilon)}(x, x) = 0, \text{ for every } x \in X \\
&d_k^{(\varepsilon)}(x_n, y_k) = d_k^{(\varepsilon)}(y_k, x_n) = \frac{\varepsilon}{n}, \text{ for } n = 1, 2, \dots \\
&d_k^{(\varepsilon)}(x_n, x_m) = \left|\frac{\varepsilon}{n} - \frac{\varepsilon}{m}\right|, \text{ for } n, m = 1, 2, \dots \\
&d_k^{(\varepsilon)}(x, y) = d_k^{(\varepsilon)}(y, x) = \varepsilon, \text{ if at least one of } x, y \text{ does not equal} \\
&\qquad\qquad\qquad\qquad \text{to } y_k \text{ or } x_n \ (n = 1, 2, \dots)
\end{aligned}$$

It is easy to show that $d_k^{(\varepsilon)} \in \mathcal{M}$ and

$$0 \leq d_k^{(\varepsilon)}(x, y) \leq \varepsilon \tag{5}$$

for every $x, y \in X$ $(k = 1, 2, \dots, 0 < \varepsilon < 1)$.

Let d be an arbitrary metric on X. Then put $\varrho_k^{(\varepsilon)} = d + d_k^{(\varepsilon)}$ for $k = 1, 2, \ldots$. According to (5) we have $\varrho_k^{(\varepsilon)} \in \overline{K}(d, \varepsilon)$ $(k \in \mathbb{N})$. Let us prove that

$$d^*(\varrho_k^{(\varepsilon)}, \varrho_l^{(\varepsilon)}) = \varepsilon \tag{6}$$

for all positive integer $k \neq l$. From (5) it follows for every $x, y \in X$, that $|d_k^{(\varepsilon)}(x,y) - d_l^{(\varepsilon)}(x,y)| \leq \varepsilon$, i.e. $d^*(d_k^{(\varepsilon)}, d_l^{(\varepsilon)}) \leq \varepsilon$.

Further we have

$$\left| d_k^{(\varepsilon)}(y_k, x_n) - d_l^{(\varepsilon)}(y_k, x_n) \right| = \left| \frac{\varepsilon}{n} - \varepsilon \right| \to \varepsilon \ (n \to \infty)$$

what implies (6) at once.

According to (6) none of the subsequences of $\{\varrho_k^{(\varepsilon)}\}_{k=1}^{\infty}$ is convergent, so the sets containing the sequence $\{\varrho_k^{(\varepsilon)}\}_{k=1}^{\infty}$ (where $0 < \varepsilon < 1$) are not compact. Since $0 < \varepsilon < 1$ was arbitrary, then it follows that $d \in \mathcal{M}$ has no open neighbourhood with compact closure. This completes the proof. □

References

[1] Frolík, Z., *Baire Spaces and Some Generalizations of Complete Metric Spaces,* Czechoslovak. Math. J. 11(86), 1961, 237-248.

[2] Sikorski, R., *Funkcje rzeczywiste I,* PWN, Warszawa, 1958.