Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 206–213

F. S. Cater, Department of Mathematics, Portland State University, Portland, Oregon 97207

On Graphs of Continuous functions

1. Throughout this note, f is a continuous real valued function on the compact interval $[a, b]$. Let l be a real number. By the set E_l, we mean the set of all real numbers c such that the intersection of the line $y = lx + c$ and the graph of f is not a perfect set. It is known (see [G2, Theorem 2′] or [BG, Theorem 4.8]) that for a typical function f in the sense of category, E_l is countable for each l. We deduce from [M] that if f has no derivative, finite or infinite, at any point, $m(E_l) = 0$ for each l. Here m denotes Lebesgue measure.

In this note we give a necessary and sufficient condition (Theorem 2.2) for $m(E_l) = 0$ for fixed l, and a necessary and sufficient condition (Theorem 2.3) for $m(E_l) = 0$ for all l. We prove that if $m(E_l) = 0$ for *two* distinct values l, then $m(E_l) = 0$ for *any* l (Theorem 2.4). This does not work when "$m(E_l) = 0$" is replaced by "E_l is a countable set" as our example F will show.

In [G1] it is proved that f is monotone on some subinterval of $[a, b]$ if and only if E_0 is a second category set. We will provide (Theorem 3.1) a condition equivalent to these properties that involves intervals on the x-axis. Finally we discuss functions of the second species (Theorem 3.2).

2. Let D be the set of all points in $[a, b]$ at which f has a derivative, finite or infinite. For any extended real number l let

$$D_l = \{x \in D : f'(x) = l\}.$$

We begin with

Lemma A. *Let l be a nonzero real number and let $g(x) = f(x) - lx$ on $[a, b]$. Then*

$$m(f(D_\infty)) = 0 \textit{ if and only if } m(g(D_\infty)) = 0, \quad \textit{and}$$

$$m(f(D_{-\infty})) = 0 \textit{ if and only if } m(g(D_{-\infty})) = 0.$$

Proof. Assume, to the contrary, that $m(f(D_\infty)) = 0$ and $m_e(g(D_\infty)) > 0$ where m_e is Lebesgue outer measure. There is a $d > 0$ such that $m_e(g(S)) > 0$

Received by the editors January 27, 1992

where

$$S = \{x \in D_\infty : (f(u) - f(x))(u-x)^{-1} > 2|l| \text{ for } 0 < |u-x| < d\}.$$

There is a subset E of S of diameter $< d$, such that $m_e(g(E)) > 0$. So $f(x) - 2|l|x$ is an increasing function on the set E.

Let h be a continuous increasing function on $[a,b]$ such that $h(x) = f(x) - 2|l|x$ on the set E. Then $h(x) + 2|l|x = f(x)$ is increasing and $h(x) + 2|l|x - lx = g(x)$ is increasing on E. Of course $m(E) = 0$ by [S, Theorem (4.4), p. 270]. We deduce from [C, Lemma 3] that either

$$m_e(g(E)) + 0 = m_e(f(E)) \text{ or } m_e(f(E)) + 0 = m_e(g(E))$$

depending on whether l is positive or negative. In either case, $m_e(f(D_\infty)) \geq m_e(f(E)) > 0$, contrary to assumption.

This contradiction proves that $m(f(D_\infty)) = 0$ implies $m(g(D_\infty)) = 0$. For the reverse implication, reverse the roles of f and g in the argument. Finally, for $D_{-\infty}$ use $-f$, $-g$ and $-l$ in place of f, g and l.

Theorem 2.1. *Let $X = \{x : x$ is an isolated point of the set $f^{-1}(f(x))\}$. Then the following conditions are equivalent.*

(1) $m(f(X)) = 0$,

(2) $m(f(D)) = 0$,

(3) $m(f(D_\infty)) = m(f(D_{-\infty})) = 0$ and $m(D \setminus D_0) = 0$.

Proof. In any case, $m(D_\infty) = m(D_{-\infty}) = 0$ by [S, Theorem (4.4), pp. 270-271].

(1) ⇒ (2) Assume (1). Every point in $D \setminus D_0$ is in X, so $m(f(D \setminus D_0)) = 0$. But $m(f(D_0)) = 0$ by [S, Theorem (4.5), p. 271]. Hence $m(f(D)) = 0$.

(2) ⇒ (3) Assume (2). Then $m(f(D_\infty)) = m(f(D_{-\infty})) = 0$ is clear. Put $S_+ = \bigcup_{0<l<\infty} D_l$ and $S_- = \bigcup_{0>l>-\infty} D_l$.

We claim that $m(S_+) = 0$. Suppose to the contrary, that $m_e(S_+) > 0$. Then there is a $c > 0$ such that $m_e(P) > 0$ where

$$P = \{x \in S_+ : f'(x) > c\}.$$

There is a $d > 0$ such that $m_e(P_1) > 0$ where

$$P_1 = \{x \in P : (f(u) - f(x))(u-x)^{-1} > 0 \text{ for } 0 < |u-x| < d\}.$$

There is a subset P_2 of P_1 of diameter $< d$ such that $m_e(P_2) > 0$. Thus f is increasing on P_2. There is a continuous increasing function h on $[a,b]$ such that

$h(x) = f(x)$ for $x \in P_2$. Now h is differentiable almost everywhere, and almost every point of P_2 is an accumulation point of P_2. It follows that $h'(x) = f'(x) > c$ almost everywhere on P_2. By [S, Lemma (9.4)(i), p. 126],

$$m_e(f(P_2)) = m_e(h(P_2)) \geq cm_e(P_2)$$

and hence

$$0 = m(f(D)) \geq cm_e(P_2),$$

so $m_e(P_2) = 0$. This contradiction proves that $m(S_+) = 0$. Likewise $m(S_-) = 0$. But $D \setminus D_0 = S_+ \cup S_- \cup D_\infty \cup D_{-\infty}$, so $m(D \setminus D_0) = 0$.

(3) $\Rightarrow$ (1) Assume (3). Then $m(X \cap (D \setminus D_0)) = 0$ and by [S, Theorem (4.5), p. 271], $m[f(X \cap (D \setminus (D_\infty \cup D_{-\infty})))] = 0$. But by (3), $m[f(X \cap D_\infty)] = m[f(X \cap D_{-\infty})] = 0$. Moreover, $D = [D \setminus (D_\infty \cup D_{-\infty})] \cup [D_\infty \cup D_{-\infty}]$, so $m(f(X \cap D)) = 0$. By [S, Lemma (6.1), p. 277], $m(f(X \setminus D)) = 0$. Finally, $m(f(X)) = 0$.

For any real number l let $f_l(x) = f(x) - lx$ on $[a, b]$.

Theorem 2.2. *Fix $l \neq 0$. Then $m(E_l) = 0$ if and only if*

$$m(D \setminus D_l) = m(f(D_\infty)) = m(f(D_{-\infty})) = 0.$$

Proof. Observe that in Theorem 2.1, $f(X) = E_0$. Moreover,

$$E_l = f_l\{x : x \text{ is an isolated point of } f_l^{-1}(f_l(x))\}.$$

These observations, together with Lemma A and Theorem 2.1, prove the conclusion.

Theorem 2.3 *We have $m(E_l) = 0$ for all real l if and only if*

$$m(D) = m(f(D_\infty)) = m(f(D_{-\infty})) = 0.$$

Proof. Let l_1 and l_2 be real numbers such that $l_1 \neq l_2$. If $m(E_{l_1}) = m(E_{l_2}) = 0$, then by Theorem 2.2, $m(D \setminus D_{l_1}) = m(D \setminus D_{l_2}) = 0$ and $m(f(D_\infty)) = m(f(D_{-\infty})) = 0$; but $D = (D \setminus D_{l_1}) \cup (D \setminus D_{l_2})$, so $m(D) = 0$.

The converse follows from Theorem 2.2.

Theorem 2.4. *Let l_1 and l_2 be real numbers such that $l_1 \neq l_2$. Let $m(E_{l_1}) = m(E_{l_2}) = 0$. Then $m(E_l) = 0$ for all real l.*

Proof. The argument is part of the proof of Theorem 2.3, so we leave it.

Proposition 2.1. *The set $\{l : E_l$ is a countable set$\}$ is a (finite or infinite) interval.*

Proof. Let $l_1 < l < l_2$ and let E_{l_1} and E_{l_2} be countable sets. Let $G(f)$ denote the graph of f and let $(x_0, f(x_0)) \in G(f)$.

Suppose that $(x_0, f(x_0))$ is an accumulation point of the intersection $G(f) \cap \{(x,y) : y - f(x_0) = l_1(x - x_0)\}$, and an accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_2(x - x_0)\}$. We consider four cases.

Case 1. $(x_0, f(x_0))$ is a left accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_i(x - x_0)\}$ for $i = 1, 2$. Then from the intermediate value theorem and the inequalities $l_1 < l < l_2$, we deduce that $(x_0, f(x_0))$ is also a left accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l(x - x_0)\}$.

Case 2. $(x_0, f(x_0))$ is a right accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_i(x - x_0)\}$ for $i = 1, 2$. Just as in case 1, we deduce that $(x_0, f(x_0))$ is also a right accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l(x - x_0)\}$.

Case 3. $(x_0, f(x_0))$ is a left accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_1(x - x_0)\}$ and a right accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_2(x - x_0)\}$. We deduce that either the function f_l has a strict local minimum point at x_0, or $(x_0, f(x_0))$ is an accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l(x - x_0)\}$.

Case 4. $(x_0, f(x_0))$ is a right accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_1(x - x_0)\}$ and a left accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l_2(x - x_0)\}$. We deduce that either the function f_l has a strict local maximum point at x_0, or $(x_0, f(x_0))$ is an accumulation point of $G(f) \cap \{(x,y) : y - f(x_0) = l(x - x_0)\}$. Of course f_l has at most countably many strict local maximum and minimum points.

But for any $c \in E_{l_i}$ $(i = 1, 2)$ there are at most countably many isolated points in $G(f) \cap \{(x,y) : y = l_i x + c\}$. From this and the assumption that $E_{l_1} \cup E_{l_2}$ is countable, we deduce that all but countably many of the points in $G(f)$ satisfy the hypothesis of the preceding paragraph. It follows that at most countably many points $(x_0, f(x_0))$ in $G(f)$ are isolated points of $G(f) \cap \{(x,y) : y - f(x_0) = l(x - x_0)\}$. Consequently E_l is a countable set.

Proposition 2.2. *Let V be a set of real numbers not bounded above or below. If E_l is countable for each $l \in V$, then E_l is countable for all real l.*

The proof follows from Proposition 2.1. The argument for Proposition 2.1 does not work when $l < l_1 < l_2$ or when $l_1 < l_2 < l$ as the following example will show.

For convenience, we call f a *P-function* if for each real l, the line $y = lx + c$ meets $G(f)$ in a perfect set for all but at most countably many c. There exist P-functions (see [BG] for example). By selecting a smaller interval domain and by adding a constant we can find a P-function f on an interval $[u, v]$ such that $f(u) = f(v) = 0$. We can find a P-function g on $[0, 1]$ with $g(0) = g(1) = 0$ by setting $g(x) = f((v - u)x + u)$. By multiplying by a constant if necessary, we assume without loss of generality that $\sup g[0,1] - \inf g[0,1] = 1$.

On any interval $[u,v]$ we define the P-function $g_{u,v}$ as follows: $g_{u,v}(x)=(v-u)g((v-u)^{-1}(x-u))$. Note that $g_{u,v}(u)=g_{u,v}(v)=0$ and $\sup g_{u,v}-\inf g_{u,v}=v-u$.

On $[0,1]$ we define the function h as follows: $h(x)=g_{\frac{1}{2},1}(x)$ for $\frac{1}{2}<x\le 1$, $h(x)=g_{\frac{1}{4},\frac{1}{2}}(x)$ for $\frac{1}{4}<x\le\frac{1}{2}$, and in general, $h(x)=g_{2^{-j-1},2^{-j}}(x)$ for $2^{-j-1}<x\le 2^{-j}$. Put $h(0)=0$. Then h is defined on $[0,1]$. Note that $G(h)$ lies above the line $y=-x$ and below the line $y=x$.

Now we make $h(x)=h(2-x)$ for $1<x\le 2$. Thus h is defined on $[0,2]$. It follows that no line not passing through the points $(0,0)$ or $(2,0)$ can meet the graphs of more than finitely many functions $g_{2^{-j-1},2^{-j}}$ and we deduce that h is a P-function on $[0,2]$. Note that $G(h)$ lies in the diamond shaped region bounded by the lines $y=x$, $y=-x$, $y=2-x$ and $y=-2+x$.

On any interval $[u,v]$, define the function $h_{u,v}(x)=\frac{1}{2}(v-u)\,h(2(v-u)^{-1}(x-u))$ for $x\in[u,v]$. It follows that $h_{u,v}$ is a P-function on $[u,v]$ whose graph lies inside the diamond shaped region bounded by the lines $y=x-u$, $y=-x+u$, $y=v-x$, $y=-v+x$. Hence:

(1) any line with slope 2 whose x-intercept is in $R\backslash[u,v]$ does not meet $G(h_{u,v})$.

Fix nonzero numbers l_1 and l_2 ($l_1\ne l_2$) such that any line of slope l_i $(i=1,2)$ whose x-intercept lies in $[0,4]$ meets $G(h)$. Hence:

(2) any line of slope l_i $(i=1,2)$ whose x-intercept lies in the interval $[u,2v-u]$ meets $G(h_{u,v})$.

Let C denote the Cantor set. For each component interval (u,v) of $(0,1)\setminus C$, put $F(x)=h_{u,v}(x)$ for $u\le x\le v$. Put $F(x)=0$ for $x\in C$. Then F is defined on $[0,1]$.

Any line of slope 2 whose x-intercept is in C must meet $G(F)$ in a singleton set by (1), not a perfect set. Thus F is not a P-function on $[0,1]$.

All but at most countably many lines of slope l_i $(i=1,2)$ whose x-intercepts lie in $R\setminus C$ meet $G(F)$ in the void set or the union of finitely many perfect sets, and therefore meet $G(F)$ in a perfect set. We deduce from (2) that all but at most countably many lines of slope l_i $(i=1,2)$ whose x-intercepts u lie in C meet $G(F)$ in the union of a sequence of perfect sets S_n such that the diameter of the set $\{(u,0)\}\cup S_n$ converges to 0. It follows that the intersection of such a line with $G(F)$ is a perfect set.

Finally, all but at most countably many lines of slope l_i $(i=1,2)$ meet $G(F)$ in a perfect set. But F is not a P-function.

3. We begin with another nuts and bolts lemma in which the function need not be everywhere continuous.

Lemma B. *Let $g(x)$ be a real valued function on $[a,b]$ such that for any points x,y,z $(x<y<z)$ in $[a,b]$,*

$$g(y) \leq \max(g(x), g(z)) \qquad \textit{and}$$

$$g(y) \geq \min(g(x), g(z)).$$

Then g is monotone on $[a,b]$.

Proof. For definiteness, say $g(a) \leq g(b)$. Take any $x, y \in [a,b]$ with $x < y$. Then

$$g(x) \leq \max(g(a), g(b)) = g(b),$$

$$g(x) \geq \min(g(a), g(b)) = g(a)$$

and so $g(a) \leq g(x) \leq g(b)$. By the same argument on x, y and b,

$$g(x) \leq g(y) \leq g(b).$$

Hence g is nondecreasing. When $g(a) \geq g(b)$ it follows analogously that g is nonincreasing.

Theorem 3.1. *Let f be a continuous function on $[a,b]$ that is constant on no interval. Let*

$$X = \{x : x \textit{ is an isolated point of } f^{-1}(f(x))\}.$$

Then the following are equivalent.

(1) *X contains a subinterval of $[a,b]$.*

(2) *X is a second category set,*

(3) *$f(X)$ contains an interval,*

(4) *$f(X)$ is a second category set,*

(5) *f is monotone on some subinterval of $[a,b]$.*

Proof. The plan is to prove (5) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (3). If I is an interval and $I \subset X$, then $f(X) \supset f(I)$, and because f is not constant on I, $f(I)$ is an interval.

(3) $\Rightarrow$ (4). Clear.

(5) $\Rightarrow$ (1). Let f be monotone on the open interval I. Then any $x \in I$ is an isolated point of $f^{-1}(f(x))$. So $I \subset X$.

(1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (4). If Y is a nowhere dense set, then $f^{-1}(Y)$ is nowhere dense; for otherwise there is an interval I in which $I \cap f^{-1}(Y)$ is dense, and $f(I \cap f^{-1}(Y))$ is also dense in the interval $f(I)$. Thus if $f(X)$ is a first category set, so are $f^{-1}(f(X))$ and X.

(4) $\Rightarrow$ (5). Let $f(X)$ be a second category set. Then there is a number $d > 0$ such that $f(W)$ is a second category set where

$$W = \{x \in X : \text{ distance from } \{x\} \text{ to } f^{-1}(f(x)) \setminus \{x\} > d\}.$$

Now (a, b) is the union of finitely many open intervals of length $< d$, so there is such an interval K such that $f(K \cap W)$ is a second category set. Note that if $x \in K \cap W$, then

$$(i) \qquad \{x\} = K \cap f^{-1}(f(x)).$$

But $f(K \cap W)$ is not nowhere dense, so there is an open interval J such that $f(K \cap W) \cap J$ is dense in J. Choose $x_0 \in K \cap W$ such that $f(x_0) \in J$. By continuity, there is an open subinterval I of K such that $f(I) \subset J$.

We claim that there exist no points $r,\ s,\ t\ (r < s < t)$ in I such that either $f(s) < \min(f(r), f(t))$ or $f(s) > \max(f(r), f(t))$. For otherwise there is a $y \in f(K \cap W) \cap J$ such that $y \in (f(s), \min(f(r), f(t))$ or $y \in (\max(f(r), f(t)), f(s))$; in either case, by the intermediate value theorem, there exist $x_1 \in (r, s)$, $x_2 \in (s, t)$ such that $y = f(x_1) = f(x_2)$, contrary to *(i)*. It follows from Lemma B that f is monotone on the interval I. (Compare with [G1, Lemma 1].)

Recall that for any real l, $f_l(x) = f(x) - lx$ on $[a, b]$.

We say that the continuous function f on $[a, b]$ is of the *second species* if for each real l, f_l is monotone on no interval. Equivalently, f is of the second species if for each integer n (positive, negative or 0), f_n is monotone on no interval.

Recall that $E_n = f_n(X_n)$ where

$$X_n = \{x : x \text{ is an isolated point of the set } f_n^{-1}(f_n(x))\}.$$

It also follows that X_n consists of those points $u \in [a, b]$ for which $(u, f(u))$ is an isolated point of $\{(x, y) : y - nx = f(u) - nu\} \cap G(f)$. Moreover, $X_0 = X$ in Theorem 3.1.

Theorem 3.2. *Let f be a continuous function on $[a, b]$ that is not linear on any interval. Then the following conditions are equivalent:*

(1) $\bigcup_{n=-\infty}^{\infty} X_n$ contains no subinterval of $[a, b]$,

(2) $\bigcup_{n=-\infty}^{\infty} X_n$ *is a first category set,*

(3) $\bigcup_{n=-\infty}^{\infty} E_n$ *contains no interval,*

(4) $\bigcup_{n=-\infty}^{\infty} E_n$ *is a first category set,*

(5) f *is of the second species.*

Proof. Note that $\bigcup_{n=-\infty}^{\infty} X_n$ (respectively $\bigcup_{n=-\infty}^{\infty} E_n$)) is a first category set if and only if each X_n (respectively E_n) is a first category set. This observation, together with Theorem 3.1, proves the result.

References

[BG] A.M. Bruckner and K.M. Garg, *The level structure of a residual set of continuous functions*, Trans. Amer. Math. Soc. 232 (1977) pp. 307-321.

[C] F.S. Cater, *Most monotone functions are not singular*, Amer. Math. Monthly 89 (1982) pp. 466-469.

[G1] K.M. Garg, *On level sets of continuous nowhere monotone functions*, Fund. Math. 52 (1963) pp. 59-68.

[G2] K.M. Garg, *On a residual set of continuous functions*, Czech. Mathematical Journal 20 (1970) pp. 537-543.

[M] S. Minakshisundaram, *On the roots of a continuous non-differentiable function*, Journal Indian Math. Soc. 4 (1940) pp. 31-33.

[S] S. Saks, *Theory of the Integral*, Second Revised Edition, Dover, New York, 1964.