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Henstock's Condition for Convergence Theorems and Equi-integrability

A sufficient condition for a sequence of Henstock–Kurzweil integrable functions to tend to an integrable limit and for the integrals of the members of the sequence to tend to the integral of the limit function can be derived using the concept of equi-integrability of a sequence of functions (see Theorem 4). From the practical point of view it is not easy to check that a given sequence of integrable functions is equi-integrable. This forces us to use another condition instead of equi-integrability and leads to the main result given in Theorem 8 which is motivated by the results of R. Henstock given in [1].

Given $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$ with $a_j < b_j$, $j = 1, \dots, n$ define an interval I (sometimes written as $[a, b]$) in $\mathbb{R}^n$ which is the set of all points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with $a_j \le x_j \le b_j$, $j = 1, \dots, n$.

$Int(I)$ is the interior of I, i.e. $Int(I)$ is the set of all $x \in \mathbb{R}^n$ with $a_j < x_j < b_j$, $j = 1, \dots, n$.

The set of all points $x \in I$ where $x_j = a_j$ or $x_j = b_j$ for exactly one $j = 1, \dots, n$ is called a *face* of I.

The boundary ∂I of I is described by those points $x \in I$ for which $x_j = a_j$ or $x_j = b_j$ for at least one $j = 1, \dots, n$. Any point $\tau \in \partial I$ belongs to at least one face of the interval I. For every face of I there is a hyperplane in $\mathbb{R}^n$ to which the face belongs. This hyperplane is parallel to one of the coordinate hyperplanes in $\mathbb{R}^n$ and splits $\mathbb{R}^n$ into two (closed) subspaces, one of them containing the interval I. For example if the face of I is given by all points $x \in I$ with $x_j = a_j$, then $\{x \in \mathbb{R}^n;\ x_j \ge a_j\}$ is the halfspace in $\mathbb{R}^n$ containing I and if the face of I is defined by $x_j = b_j$, then $\{x \in \mathbb{R}^n;\ x_j \le b_j\}$ is the halfspace containing I.

Assume that $I = [a, b]$ is an interval in $\mathbb{R}^n$ and $\tau \in \mathbb{R}^n$ is a given point. Define the *extension* $E_\tau(I)$ of I with respect to the point τ as follows.

a) If $\tau \notin I$, then $E_\tau(I) = \emptyset$

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a) If $\tau \notin I$, then $E_\tau(I) = \emptyset$

b) If $\tau \in Int(I)$, then $E_\tau(I) = \mathbb{R}^n$

c) If $\tau \in \partial I$, then $E_\tau(I)$ is the intersection of all halfspaces containing I where the hyperplanes defining these halfspaces are given by all the faces to which the point τ belongs.

Let an interval $I = [a, b] \subset \mathbb{R}^n$ be given. A pair (τ, J) of a point $\tau \in \mathbb{R}^n$ and a compact interval $J \subset \mathbb{R}^n$ is called a *tagged interval*, τ is the *tag* of J.

A finite collection $\Delta = \{(\tau_j, J_j),\ j = 1, \dots, k\}$ of tagged intervals is called a *system* in I if $\tau_j \in J_j \subset I$ for every $j = 1, \dots, k$ and the intervals J_j are nonoverlapping, i.e. $Int(J_i) \cap Int(J_j) = \emptyset$ for $i \neq j$.

A system $\Delta = \{(\tau_j, J_j),\ j = 1, \dots, k\}$ is called a *partition* of I if

$$\bigcup_{j=1}^{k} J_j = I.$$

Given a positive function $\delta : I \to (0, +\infty)$ called a *gauge* on I, a tagged interval (τ, J) with $\tau \in [a, b]$ is said to be *δ–fine* if

$$J \subset B(\tau, \delta(\tau)) = \{x \in \mathbb{R}^n; \|x - \tau\| \leq \delta(\tau)\}.$$

A *system* (in particular, a *partition*) $\Delta = \{(\tau_j, J_j), j = 1, \dots, k\}$ is δ–fine if the point–interval pair (τ_j, J_j) is δ–fine for every $j = 1, \dots, k$.

Definition 1 *Let $h(\tau, J)$ be a finite real-valued function of point–interval pairs with $\tau \in I$, $J \subset I$. The function h is called integrable over I if there is an $H \in \mathbb{R}$ such that given $\varepsilon > 0$, there is a gauge δ on I such that*

$$\left| \sum_{j=1}^{k} h(\tau_j, J_j) - H \right| < \varepsilon$$

for every δ–fine partition $\Delta = \{(\tau_j, J_j),\ j = 1, \dots, k\}$ of I. The number H is called the integral of the function h over I and it is denoted $\int_I dh$. We denote by $\mathcal{K}(I) = \mathcal{K}([a, b])$ the set of all functions h which are integrable over $I = [a, b]$.

This is the definition of the *Henstock–Kurzweil integral (generalized Riemann integral)* that can be found in a still growing series of books on integration (see e.g. [1], [2], [3], [4], [5], [6], [7]) where this idea of integration is explained from various aspects.

For the Riemann sum $\sum_{j=1}^{k} h(\tau_j, J_j)$ which corresponds to the function h and to a given partition Δ of I let us denote

$$S(h, \Delta) = \sum_{j=1}^{k} h(\tau_j, J_j).$$

Clearly Definition 1 is viable only if for a given gauge δ on I there exists at least one δ–fine partition D of I. This fundamental question has an affirmative answer given by the following statement.

Lemma 1 (Cousin) *Given a gauge δ on I, there is a δ–fine partition $\Delta = \{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of I.*

For the proof of this lemma see e.g.[1, Theorem 4.1].

The following statement provides an operative tool in the theory of generalized Perron integral. Its original version belongs to S. Saks and it was formulated for generalized integrals using Riemann–like sums by R. Henstock.

Lemma 2 (Saks–Henstock) *Let $h : I \to \mathbb{R}$ be integrable over the interval $I \subset \mathbb{R}^n$. Given $\varepsilon > 0$ assume that the gauge δ on I is such that*

$$\left| \sum_{j=1}^{k} h(\tau_j, J_j) - \int_I dh \right| < \varepsilon$$

for every δ–fine partition $\Delta = \{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of I. If D is a δ–fine system $\{(\xi_j, L_j),\ j = 1, \ldots, m\}$, i.e.

$$\xi_j \in L_j \subset B(\xi_j, \delta(\xi_j)), \quad j = 1, \ldots, m,$$

then

$$\left| \sum_{j=1}^{k} \left[h(\xi_j, L_j) - \int_{L_j} dh \right] \right| \leq \varepsilon \tag{1}$$

See e.g. [1], Theorem 5.3.

Theorem 3 *Let real valued point–interval functions $h, h_m, m = 1, 2, \ldots$ be given where $h_m \in \mathcal{K}(I)$ for $m = 1, 2, \ldots$. Assume that there is a gauge ω on I such that for every ω–fine pair (τ, J), $(\tau \in I)$ we have*

$$\lim_{m \to \infty} h_m(\tau, J) = h(\tau, J).label2 \tag{2}$$

Assume further that

for every $\eta > 0$ there is a gauge δ on I such that

$$(3) \qquad |S(h_m, D) - \int_I dh_m| < \eta$$

for every δ-fine partition D of I and every $m = 1, 2, \ldots$.

Then $h \in \mathcal{K}(I)$ and

$$(4) \qquad \lim_{m\to\infty} \int_I dh_m = \int_I dh.$$

Proof. Let $\varepsilon > 0$ be given. By (3) there is a gauge δ on I, $\delta(\tau) \leq \omega(\tau)$, $\tau \in I$ such that for every δ-fine partition $D = \{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of I we have

$$|S(h_m, D) - \int_I dh_m| < \frac{\varepsilon}{2}$$

for $m = 1, 2, \ldots$. By (2) for every fixed partition D of I there exists a positive integer m_0 such that for $m > m_0$ the inequality

$$|S(h_m, D) - S(h, D)| = |\sum_{j=1}^{k} [h_m(\tau_j, J_j) - h(\tau_j, J_j)]| < \frac{\varepsilon}{2}$$

holds and this means that

$$\lim_{m\to\infty} S(h_m, D) = S(h, D).$$

Therefore for any δ-fine partition D of I there is a positive integer m_0 such that for $m > m_0$ we have

$$(5) \qquad |S(h, D) - \int_I dh_m| < \varepsilon.$$

First we get from (5) that for all positive integers $m, l > m_0$ the inequality

$$\left|\int_I dh_m - \int_I dh_l\right| < 2\varepsilon$$

holds. This means that $(\int_I dh_m)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ and it has therefore a limit

$$(6) \qquad \lim_{m\to\infty} \int_I dh_m = H \in \mathbb{R}.$$

The second consequence of (5) is the inequality

$$|S(h, D) - H| \leq$$

$$|S(h, D) - \int_I dh_m| + |\int_I dh_m - H| < \varepsilon + |\int_I dh_m - H|.$$

By (6) we obtain immediately from this inequality that for every δ-fine partition D of I we have $|S(h, D) - H| < \varepsilon$ and this means that the integral $\int_I dh$ exists and that (4) is satisfied.

Definition 2 *A sequence of real valued point-interval functions h_m, $m = 1, 2 \ldots$ which are integrable over I ($h_m \in \mathcal{K}(I)$) is called equi-integrable if condition (3) of Theorem 4 is satisfied.*

Remark 1 Theorem 4 gives a sufficient condition for a sequence of integrable functions to tend to an integrable limit and for the integrals of the members of the sequence to tend to the integral of the limit function. The convergence of the functions h_m to h is given by (2) and the sufficient condition is the equi-integrability (3) of the sequence (h_m).

The fundamental idea of the proof of Theorem 4 lies in understanding the concept of the integral as a certain limiting process and in the fact that two limits are interchangeable provided one of them is uniform with respect to the limiting variable of the second. In our situation equi-integrability stands for this uniformity. (See also [6] and [9].) In this sense Theorem 4 is a transparent mathematical fact. Nevertheless from the practical point of view it is not easy to check that a given sequence of integrable functions is equi-integrable.

This forces us to use another condition instead of equi-integrability and motivates the results given in the sequel. The forthcoming Theorem 8 is motivated by the results given in Theorem 9.1 in [1].

Definition 3 *A real valued function h of point-interval pairs is called interval additive if $h(\tau, J) = h(\tau, J_1) + h(\tau, J_2)$ provided J, J_1, J_2 are intervals such that $J = J_1 \cup J_2$ and $Int(J_1) \cap Int(J_2) = \emptyset$ and $\tau \in J_1 \cap J_2$.*

For example the real valued point-interval function h defined by

$$h(\tau, J) = v(J) = \prod_{j=1}^{n} (d_j - c_j)$$

for an interval $J = [c, d]$ with $c, d \in \mathbb{R}^n$, $c_j < d_j$, $j = 1, \ldots, n$ and an arbitrary $\tau \in \mathbb{R}^n$ is interval additive.

Theorem 4 *Let real valued interval additive point-interval functions h, h_m, $m = 1, 2, \ldots$ be given where $h_m \in \mathcal{K}(I)$ for $m = 1, 2, \ldots$.*

Assume that there is a gauge ω on I such that for every $\varepsilon > 0$ there exist a $p : I \to \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers.) and a positive superadditive interval function Φ defined for closed intervals $J \subset I$ with $\Phi(I) < \varepsilon$ such that for every $\tau \in I$ we have

$$|h_m(\tau, J) - h(\tau, J)| < \Phi(J) \tag{7}$$

provided $m > p(\tau)$ and (τ, J) is an ω-fine tagged interval with $\tau \in J \subset I$.

Let us assume further that the sequence (h_m) satisfies the following condition.

There exist a gauge θ on I and real constants $B < C$ such that for all choices of functions m defined on I taking positive integer values ($m : I \to \mathbb{N}$) the inequalities

$$B \le \sum_{j=1}^{k} h_{m(\tau_j)}(\tau_j, J_j) \le C \tag{8}$$

hold provided $D = \{(\tau_j, J_j), j = 1, 2, \ldots, k\}$ is an arbitrary θ-fine partition of I.

Then the sequence (h_m) is equi-integrable; i.e. (3) holds.

Let us first give the following definition.

Definition 4 *Let real valued interval additive point-interval functions h, h_m, $m = 1, 2, \ldots$, $h_m \in \mathcal{K}(I)$ for $m = 1, 2, \ldots$ be given such that the conditions (7) and (8) of Theorem 8 are satisfied.*

For a given $p \in \mathbb{N}$ let $\mathcal{S}_p$ be the family of all real valued point-interval functions v such that there is a finite system of nonoverlapping intervals $L_j, j = 1, \ldots, l$ in I with $\bigcup_{j=1}^{l} L_j = I$ such that for a tagged interval (τ, J), $\tau \in J \subset I$ we have

$$v(\tau, J) = \sum_{j=1}^{m} h_{m_j}(\tau, J \cap E_\tau(L_j)) \tag{9}$$

where $m_j \in \mathbb{N}$ with $m_j \ge p, j = 1, 2, \ldots, l$.

Looking at the definition of the extension $E_\tau(L_j)$ of the interval L_j with respect to the point τ we see immediately that in (9) $v(\tau, J) = \sum_{j=1}^{m} h_{m_j}(\tau, J \cap E_\tau(L_j)) = h_{m_j}(\tau, J \cap E_\tau(L_j)) = h_{m_j}(\tau, J)$ if $\tau \in Int(L_j)$ for some j. If $\tau \notin \bigcup_j Int(L_j)$, then τ belongs to the boundaries of some intervals L_j and $v(\tau, J)$ is the sum of values of all point-interval functions $h_{m_j}(\tau, J \cap E_\tau(L_j))$ for which the point τ belongs to L_j and the interval $J \cap E_\tau(L_j)$ is the portion of J which belongs to the extension $E_\tau(L_j)$.

Now we give a series of statements concerning the families $\mathcal{S}_p$ given by Definition 9.

Lemma 5 *a) For every $p \in \mathbb{N}$ we have $h_p \in \mathcal{S}_p$.*

b) If $p_1, p_2 \in \mathbb{N}$, $p_1 > p_2$, then $\mathcal{S}_{p_1} \subset \mathcal{S}_{p_2}$.

c) If $v \in \mathcal{S}_1$, then $v \in \mathcal{K}(I)$.

Proof. a) If we set $L_1 = I$, then clearly h_p is of the form (9).

b) is clear by the definition of $\mathcal{S}_p$.

c) Given $v \in \mathcal{S}_p$ it is easy to check by (9) that for every partition D^j of L_j we have

$$S(v, D^j) = S(h_{m_j}, D^j)$$

for the corresponding integral sums. Since h_{m_j} is integrable over I, it is also integrable over L_j and (10) yields the integrability of v over L_j. This holds for every $j = 1, \ldots, l$ and therefore we get $v \in \mathcal{K}(I)$.

Lemma 6 *If $v \in \mathcal{S}_1$, then*

$$B \leq \sum_{j=1}^{k} v(\tau_j, J_j) = S(v, D) \leq C \tag{10}$$

for an arbitrary θ-fine partition $D = \{(\tau_j, J_j),\ j = 1, 2, \ldots, k\}$ of I.

Proof. Let $\{L_1, \ldots, L_m\}$ be the finite sequence of intervals used for the definition of v in (9). It is easy to see that if $D = \{(\tau_j, J_j),\ j = 1, 2, \ldots, k\}$ is an arbitrary θ-fine partition of I, then $\{(\tau_j, J_j \cap E_{\tau_j}(L_i)), j = 1, \ldots, k, i = 1, \ldots, l\}$ is also a θ-fine partition of I and therefore (11) follows immediately from (8).

Lemma 7 *If (τ, J) is an ω-fine tagged interval, $\tau \in J \subset I$, $p \geq p(\tau)$, then*

$$|v(\tau, J) - h(\tau, J)| < \Phi(J) \tag{11}$$

for every $v \in \mathcal{S}_p$.

Proof. If $v \in \mathcal{S}_p$, then by Definition 9 there is a finite system of nonoverlapping intervals $L_j, j = 1, \ldots, l$ in I, $\bigcup_{j=1}^{l} L_j = I$ such that for the ω-fine tagged interval (τ, J), $\tau \in J \subset I$ we have $v(\tau, J) = \sum_{j=1}^{l} h_{m_j}(\tau, J \cap E_\tau(L_j))$ with $m_j \in \mathbb{N}$, $m_j \geq p$, for $j = 1, 2, \ldots, l$. Since $\tau \in J \cap E_\tau(L_j)$ when $J \cap E_\tau(L_j) \neq \emptyset$, the tagged intervals $(\tau, J \cap E_\tau(L_j))$ are ω-fine for every j and $\bigcup_j (J \cap E_\tau(L_j) = J$ we have by the interval additivity of h and by (7)

$$|v(\tau, J) - h(\tau, J)| = |\sum_{j=1}^{m} [h_{m_j}(\tau, J \cap E_\tau(L_j)) - h(\tau, J \cap E_\tau(L_j))]| \leq$$

$$\leq \sum_{j=1}^{m} |h_{m_j}(\tau, J \cap E_\tau(L_j)) - h(\tau, J \cap E_\tau(L_j))| \leq$$

$$\leq \sum_{j=1}^{m} \Phi(J \cap E_\tau(L_j)) \leq \Phi(J)$$

because Φ is a superadditive interval function.

Lemma 8 *Assume that $\varepsilon > 0$. Then to every $p \in \mathbb{N}$ there exist $v_p, v^p \in \mathcal{S}_p$ such that*

$$(12)\int_I dv_p - \frac{\varepsilon}{2^p} < \inf\left\{\int_I dv; v \in \mathcal{S}_p\right\} \leq \sup\left\{\int_I dv; v \in \mathcal{S}_p\right\} < \int_I dv^p + \frac{\varepsilon}{2^p}.$$

Proof. By Lemma 11 the supremum and infimum exist and

$$B \leq \inf\left\{\int_I dv;\ v \in \mathcal{S}_p\right\} \leq \sup\left\{\int_I dv;\ v \in \mathcal{S}_p\right\} \leq C.$$

The existence of v_p, v^p satisfying (13) follows immediately from the definition of the infimum and supremum.

Lemma 9 *Assume that $v \in \mathcal{S}_p$. Let I_j, $j = 1, \ldots, s$ be an arbitrary finite sequence of closed nonoverlapping intervals in I. Then for a given $\varepsilon > 0$ we have*

$$\sum_{j=1}^{s} \int_{I_j} dv_p - \frac{\varepsilon}{2^p} \leq \sum_{j=1}^{s} \int_{I_j} dv \leq \sum_{j=1}^{s} \int_{I_j} dv^p + \frac{\varepsilon}{2^p}$$

where $v_p, v^p \in \mathcal{S}_p$ are the functions corresponding to ε by Lemma 13.

Proof. Assume for example that the second inequality in (14) is not satisfied. Then there is a $v^* \in \mathcal{S}_p$ such that $\sum_{j=1}^{s} \int_{I_j} dv^p + \frac{\varepsilon}{2^p} < \sum_{j=1}^{s} \int_{I_j} dv^*$. For the system of intervals $\{I_j,\ j = 1, \ldots, s\}$ there exists a finite system $\{L_k,\ k = 1, \ldots, r\}$ of closed intervals in I such that both systems together form a system of nonoverlapping intervals with $\bigcup_j I_j \cup \bigcup_k L_k = I$. Set

$$v_0(\tau, J) = \sum_{j=1}^{s} v^*(\tau, J \cap E_\tau(I_j)) + \sum_{k=1}^{r} v^p(\tau, J \cap E_\tau(L_k)).$$

It is not difficult to check that $v_0 \in \mathcal{S}_p$ and we have the inequality

$$\int_I dv_0 = \sum_{j=1}^{s} \int_{I_j} dv^* + \sum_{k=1}^{r} \int_{L_k} dv^p >$$

$$> \sum_{j=1}^{s} \int_{I_j} dv^p + \sum_{k=1}^{r} \int_{L_k} dv^p + \frac{\varepsilon}{2^p} = \int_I dv^p + \frac{\varepsilon}{2^p}$$

which contradicts Lemma 13 and therefore proves the second inequality in (14). The first inequality in (14) holds by a similar argument.

Proof of Theorem 8 Let $\varepsilon > 0$ be given. By (7) and Lemma ?? for every $\tau \in I$ there is a $p(\tau) \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m \geq p(\tau)$ we have

$$|v_{p(\tau)}(\tau, J) - h_m(\tau, J)| \leq \\ |v_{p(\tau)}(\tau, J) - h(\tau, J)| + |h_m(\tau)(\tau, J) - h(\tau, J)| \leq 2\Phi(J) \tag{13}$$

and similarly also

$$|v^{p(\tau)}(\tau, J) - h_m(\tau, J)| \leq 2\Phi(J) \tag{14}$$

if (τ, J) is an ω-fine tagged pair with $\tau \in J \subset I$ where Φ is a positive superaditive interval function with $\Phi(I) < \varepsilon$.

For a given $p \in \mathbb{N}$ the functions h_p, v_p, v^p are integrable and therefore there is a gauge δ_p on I such that

$$|S(h_p, D) - \int_I dh_p| < \frac{\varepsilon}{2^p}$$

$$|S(v_p, D) - \int_I dv_p| < \frac{\varepsilon}{2^p} \tag{15}$$

$$|S(v^p, D) - \int_I dv^p| < \frac{\varepsilon}{2^p}$$

for any δ_p- fine partition D of I. For $\tau \in I$ let us choose $\delta(\tau) > 0$ such that

$$\delta(\tau) \leq \min(\omega(\tau), \delta_1(\tau), \delta_2(\tau), \ldots, \delta_{p(\tau)}(\tau))$$

(ω is the gauge from (7) and (15), $p : I \to \mathbb{N}$ is given in Lemma 12 and the gauges $\delta_1, \delta_2, \ldots$ come from (17)).

Assume that $D = \{(\tau_i, J_i), i = 1, \ldots, k\}$ is a δ-fine partition of I and that $m \in \mathbb{N}$ is arbitrary. Then

$$S(h_m, D) = \sum_{\substack{i=1 \\ m \leq p(\tau_i)}}^{k} h_m(\tau_i, J_i) + \sum_{\substack{i=1 \\ m > p(\tau_i)}}^{k} h_m(\tau_i, J_i). \tag{16}$$

The first term is the sum of those $h_m(\tau_i, J_i)$ for which $m \leq p(\tau_i)$ and similarly for the second term. We show that $S(h_m, D) > \int_I dh_m - 4\varepsilon$.

If $m > p(\tau_i)$, then by (16) $-2\Phi(J) \leq h_m(\tau_i, J_i) - v^{p(\tau_i)}(\tau_i, J_i) \leq 2\Phi(J)$. Therefore $h_m(\tau_i, J_i) \geq v^{p(\tau_i)}(\tau_i, J_i) - 2\Phi(J)$ and for the second sum in (18) we have

$$\sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} h_m(\tau_i, J_i) \geq \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} [v^{p(\tau_i)}(\tau_i, J_i) - 2\Phi(J)].$$

Hence

$$S(h_m, D) \geq \sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} h_m(\tau_i, J_i) + \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} v^{p(\tau_i)}(\tau_i, J_i) - 2 \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \Phi(J) =$$

$$= -2 \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \Phi(J) + \sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} h_m(\tau_i, J_i) + \sum_{l=1}^{m-1} \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} v^{p(\tau_i)}(\tau_i, J_i).$$

From (17) and from the Saks–Henstock Lemma 3 we obtain

$$\left| \sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} h_m(\tau_i, J_i) - \sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} \int_{J_i} dh_m \right| < \frac{\varepsilon}{2^m}$$

and

$$\left| \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} v^l(\tau_i, J_i) - \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dv^l \right| < \frac{\varepsilon}{2^l}.$$

That is,

$$\sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} h_m(\tau_i, J_i) > \sum_{\substack{i=1\\ m\leq p(\tau_i)}}^{k} \int_{J_i} dh_m - \frac{\varepsilon}{2^m}$$

and

$$\sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} v^l(\tau_i, J_i) > \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dv^l - \frac{\varepsilon}{2^l}.$$

Thus

$$S(h_m,D) > -2 \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \Phi(J) + \sum_{\substack{i=1\\ m\le p(\tau_i)}}^{k} \int_{J_i} dh_m - \frac{\varepsilon}{2^m} +$$

$$+ \sum_{l=1}^{m-1} \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dv^l - \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l}.$$

Since $h_m \in \mathcal{S}_m$, we have by b) of Lemma 10 that $h_m \in \mathcal{S}_p$ for all $p = 1, 2, \ldots, m-1$ and by Lemma 14 we therefore get

$$\sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dv^l \ge \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dh_m - \frac{\varepsilon}{2^l}$$

i.e.

$$\sum_{l=1}^{m-1} \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dv^l \ge \sum_{l=1}^{m-1} \sum_{\substack{i=1\\ p(\tau_i)=l}}^{k} \int_{J_i} dh_m - \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l} =$$

$$= \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \int_{J_i} dh_m - \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l}$$

and

$$S(h_m,D) > -2 \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \Phi(J) - \frac{\varepsilon}{2^m} - \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l} - \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l} +$$

$$+ \sum_{\substack{i=1\\ m>p(\tau_i)}}^{k} \int_{J_i} dh_m + \sum_{\substack{i=1\\ m\le p(\tau_i)}}^{k} \int_{J_i} dh_m \ge$$

$$\ge -2\Phi(I) - \frac{\varepsilon}{2^m} - 2 \sum_{l=1}^{m-1} \frac{\varepsilon}{2^l} + \int_I dh_m \ge$$

$$\ge \int_I dh_m - \varepsilon\Big(2 + \frac{1}{2^m} + 2 \sum_{l=1}^{m-1} \frac{1}{2^l}\Big) > \int_I dh_m - 4\varepsilon.$$

In a completely analogous way we can use v_p instead of v^p to show that

$$S(h_m, D) < \int_I dh_m + 4\varepsilon$$

i.e.

$$|S(h_m, D) - \int_I dh_m| < 4\varepsilon$$

and the sequence (h_m) is equi-integrable with the gauge δ being independent of m.

Corollary 10 *If conditions (7) and (8) of Theorem 8 are satisfied for the functions $h_m, h, m = 1, 2, \ldots$, then $h \in \mathcal{K}(I)$ and*

$$\lim_{m\to\infty} \int_I dh_m = \int_I dh.$$

Proof. The result follows immediately from Theorems 8 and 4.

Lemma 11 *Assume that h_m, $m = 1, 2, \ldots$ are point-interval functions defined for tagged intervals (τ, J), $\tau \in J \subset I$. Let $v, w \in \mathcal{K}(I)$ be point-interval functions such that*

$$v(\tau, J) \le h_m(\tau, J) \le w(\tau, J) \tag{17}$$

for every tagged interval (τ, J) and $m \in \mathbb{N}$. Then the sequence (h_m) satisfies condition (8) of Theorem 8.

Proof. Since $v, w \in \mathcal{K}(I)$, there is a gauge θ on I such that

$$|S(v, D) - \int_I dv| < 1 \tag{18}$$

and

$$|S(w, D) - \int_I dv| < 1 \tag{19}$$

provided $D = \{(\tau_j, J_j),\ j = 1, \ldots, k\}$ is a θ-fine partition of I.

Let $m : I \to \mathbb{N}$ be arbitrary. Then by (19) $v(\tau_j, J_j) \le h_{m(\tau_j)}(\tau_j, J_j) \le w(\tau_j, J_j)$ for every tagged interval (τ_j, J_j) belonging to a θ-fine partition D of I and

$$S(v, D) \le \sum_{j=1}^{k} h_{m(\tau_j)}(\tau_j, J_j) \le S(w, D).$$

By (20) and (21) we have $B = \int_I dv - 1 < S(v, D)$ and $S(w, D) < \int_I dw + 1 = C$ and therefore (8) of Theorem 8 is satisfied.

Corollary 12 (The dominated convergence theorem) *Let real valued interval additive point–interval functions h, h_m, $m = 1, 2, \ldots$ be given where $h_m \in \mathcal{K}(I)$ for $m = 1, 2, \ldots$.*

Assume that there is a gauge ω on I such that for every $\varepsilon > 0$ there exist a $p : I \to \mathbb{N}$ and a positive superadditive interval function Φ defined for closed intervals $J \subset I$ with $\Phi(I) < \varepsilon$ such that for every $\tau \in I$ we have

$$|h_m(\tau, J) - h(\tau, J)| < \Phi(J) \tag{7}$$

provided $m > p(\tau)$ and (τ, J) is an ω-fine tagged interval with $\tau \in J \subset I$.

Let us assume further that the sequence (h_m) satisfies the following condition.

There exist $v, w \in \mathcal{K}(I)$ point–interval functions such that

$$v(\tau, J) \leq h_m(\tau, J) \leq w(\tau, J) \tag{19}$$

for every tagged interval (τ, J), $(\tau \in J \subset I)$ and every $m \in \mathbb{N}$.

Then the sequence (h_m) is equi-integrable, h is integrable over I and

$$\lim_{m \to \infty} \int_I dh_m = \int_I dh.$$

Proof. The result follows from Lemma 16 and Corollary 15.

If an interval $I \subset \mathbb{R}^n$ is given and $f : I \to \mathbb{R}$ is a point function, then we set $h(\tau, J) = f(\tau).v(J)$ for tagged intervals (τ, J), $(\tau \in J \subset I)$, where $v(J)$ is the volume (Lebesgue measure) of the interval J and we write

$$\int_I f dv = \int_I dh$$

provided the integral on the right hand side exists. This is the Henstock–Kurzweil integral of a point function which is equivalent to the classical Perron integral in the case $n = 1$. From the results given above for this special case the following theorem can be deduced.

Corollary 13 *Let real valued point functions $f, f_m : I \to \mathbb{R}$, $m = 1, 2, \ldots$ be given where the integrals $\int_I f_m dv$ exist for $m = 1, 2, \ldots$. Assume that*

$$\lim_{m \to \infty} f_m(\tau) = f(\tau) \tag{22}$$

for every $\tau \in I$ and that there is a gauge θ on I and real constants $B < C$ such that for all choices of functions $m : I \to \mathbb{N}$ the inequalities

$$B \leq \sum_{j=1}^{k} f_{m(\tau_j)}(\tau_j) v(J_j) \leq C \tag{23}$$

hold provided $D = \{(\tau_j, J_j), j = 1, \dots, k\}$ *is an arbitrary* θ*-fine partition of* I. *Then the sequence of point-interval functions* $h_m(\tau, J) = f_m(\tau).v(J)$ *is equi-integrable, the integral* $\int_I f dv$ *exists and*

$$\lim_{m\to\infty} \int_I f_m dv = \int_I f dv.$$

Proof. Let us set $h(\tau, J) = f(\tau).v(J)$. Then

$$|h_m(\tau, J) - h(\tau, J)| = |\left[f_m(\tau) - f(\tau)\right].v(J)| \leq |f_m(\tau) - f(\tau)|.v(J).$$

By (22) for every $\varepsilon > 0$ there is a $p : I \to \mathbb{N}$ such that for every $\tau \in I$ we have

$$|f_m(\tau) - f(\tau)| < \frac{\varepsilon}{v(I) + 1}$$

provided $m > p(\tau)$. Hence

$$|h_m(\tau, J) - h(\tau, J)| < \frac{\varepsilon}{v(I) + 1}.v(J)$$

and condition (7) of Theorem 8 is satisfied in this case with the additive interval function $\Phi(J) = \frac{\varepsilon}{v(I)+1}.v(J)$ for which $\Phi(I) = \frac{\varepsilon}{v(I)+1}.v(I) < \varepsilon$.

Theorem 8 yields the equi-integrability of the sequence h_m and the convergence result comes from Corollary 15.

Remark 2 Condition (23) in Corollary 18 is the condition given by R. Henstock in [1, Theorem 9.1]. The same can also be said about condition (8) in Theorem 8.

See [3, 5.4. Lemma] for a result that is similar to Theorem 8. There the following condition is introduced.

> *There is a constant* $K > 0$ *such that for every finite system of nonoverlapping intervals* $L_j, j = 1, \dots, l$ *in* I *with* $\bigcup_{j=1}^{l} L_j = I$ *and every finite sequence* $m_1, \dots, m_l \in \mathbb{N}$ *the inequality*
>
> $$\left|\sum_{j=1}^{l} \int_{L_j} dh_{m_j}\right| \leq K$$
>
> *holds.*

It is shown in [3] that this condition together with (7) also implies the equi-integrability of the sequence (h_m). It is worth mentioning that the idea of conditions of this type as well as the method of proving a theorem like our Theorem 8 goes back to the mimeographed notes [8] written in 1979. These notes form a draft version of a chapter on integration of a textbook on calculus which was not published.

Example 1 *The following simple example shows that equi-integrability of a sequence does not imply Henstock's condition (e.g. condition (23) in Corollary 18).*

Assume that $I \subset \mathbb{R}^n$ is an interval and that $g : I \to \mathbb{R}$ is such that $g \in \mathcal{K}(I)$ and $|g| \notin \mathcal{K}(I)$. Define $f_m(\tau) = \frac{1}{m} g(\tau)$ for $\tau \in I$. We evidently have $f_m \in \mathcal{K}(I)$, $f_m(\tau) \to 0$ and $\int_I f_m dv = \frac{1}{m} \int_I g dv \to 0$ for $m \to \infty$. Suppose that there is a gauge θ on I and real constants $B < C$ such that for all choices of functions $m : I \to \mathbb{N}$ the inequalities

$$B \le \sum_{j=1}^{k} f_{m(\tau_j)}(\tau_j) v(J_j) \le C \tag{23}$$

hold provided $D = \{(\tau_j, J_j),\ j = 1, \dots, k\}$ is an arbitrary θ-fine partition of I.

For $m(\tau) = 1$ for $\tau \in I$ we have

$$B \le \sum_{j=1}^{k} f_1(\tau_j) v(J_j) \le C \tag{24}$$

for every θ-fine partition D of I. Putting $m(\tau) = m$ when $g(\tau) \ge 0$ and $m(\tau) = 1$ otherwise we obtain

$$B - C \le \sum_{j=1}^{k} \left[f_1(\tau_j) - f_{m(\tau_j)}(\tau_j) \right] v(J_j) \le C - B$$

and by the construction of the function m we have $f_1(\tau_j) - f_{m(\tau_j)}(\tau_j) = (1 - \frac{1}{m}) g(\tau_j)$ if $g(\tau_j) \ge 0$ and $f_1(\tau_j) - f_{m(\tau_j)}(\tau_j) = 0$ if $g(\tau_j) = 0$ i.e.

$$B - C \le \sum_{\substack{j=1 \\ g(\tau_j) \ge 0}}^{k} (1 - \frac{1}{m}) g(\tau_j) v(J_j) \le C - B.$$

Similarly if $m(\tau) = m$ when $g(\tau) < 0$ and $m(\tau) = 1$ otherwise we obtain

$$B - C \le \sum_{\substack{j=1 \\ g(\tau_j) < 0}}^{k} (1 - \frac{1}{m}) g(\tau_j) v(J_j) \le C - B.$$

Therefore

$$\sum_{j=1}^{k} (1 - \frac{1}{m}) |g(\tau_j)| v(J_j) \le 2(C - B)$$

and we have the inequality

$$\sum_{j=1}^{k} |g(\tau_j)|v(J_j) \leq \frac{2m}{m-1}(C-B)$$

which contradicts the fact that g is not absolutely integrable ($|g| \notin \mathcal{K}(I)$). The sequence (f_m) does not satisfy Henstock's condition (23) (or (8)) and Theorem 8 does not contain the convergence result which holds for this example.

On the other hand it is easy to see that the sequence (f_m) is equi-integrable and the convergence result is guaranteed by Theorem 4.

References

[1] R. Henstock *Lectures on the Theory of Integration*, World Scientific, Singapore, 1988

[2] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, 1991

[3] J. Kurzweil, *Nichtabsolut konvergente Integrale*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980

[4] Lee Peng-Yee , *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989

[5] J. Mawhin, *Introduction a l'Analyse*, CABAY, Libraire- diteur, Louvain–La–Neuve, 1984

[6] R. M. McLeod, *The Generalized Riemann Integral*, Math. Assoc. of America, 1980

[7] E. J. McShane, *Unified Integration*, Academic Press, New York, London, 1983

[8] D. Preiss, Š. Schwabi, *The definite integral*, Mimeographed notes in Czech, Math.-Phys. Fac. of the Charles Univ. and Math. Inst. of the Czech. Acad. Sci, Prague, 1979

[9] B. S. Thomson, *Derivation bases on the real line (II)*, Real Analysis Exchange, 8 (1982–83), 278–442