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# Derivatives, Continuous Functions and Bounded Lebesgue Functions

It is well-known that the product of a derivative with, say, a continuous function need not be a derivative. This fact leads naturally to the following problem: Let $\Omega$ be a class of derivatives. Characterize the multipliers of $\Omega$, i.e. the functions $f$ such that $fg$ is a derivative for each $g \in \Omega$. This problem has been solved for various classes $\Omega$. (See [F1], [F2], [M1], ... , [M4].) In this note we describe the multipliers, first, of continuous functions (theorem 7) and, second, of bounded Lebesgue functions (theorem 12); these multipliers play an important role in theorem 5.11 in [MW]. It is well-known that the class of bounded Lebesgue functions is identical with the class of bounded approximately continuous functions. (The multipliers of all Lebesgue functions are just the bounded derivatives; see theorem 4.2 in [M4].)

**1. Notation.** We write, as usual, $\mathbb{R} = (-\infty, \infty)$. The word function means a mapping to $\mathbb{R}$. Further we set $\mathbb{R}^+ = (0, \infty)$, $I = [0, 1]$. The symbol $D$ stands for the system of all finite derivatives on $I$. The words measure and measurable refer to Lebesgue measure in $\mathbb{R}$; the measure of a measurable set $S \subset \mathbb{R}$ will be denoted by $|S|$. If $x \in \mathbb{R}$ and if $S$ is a measurable subset of $\mathbb{R}$, then $d(S, x)$ is defined as $\lim(|S \cap (x - h, x + h)|/2h)$ $(h \to 0+)$ provided that the limit exists.

Symbols like $\int_a^b f$ or $\int_S f$ mean the corresponding Perron or Lebesgue integrals; "integrable" means "Perron integrable". (We need an integral that integrates every derivative.) If $a > b$, then, as usual, we set $\int_a^b f = -\int_b^a f$ provided that the last integral exists.

If $J$ is an open interval in $\mathbb{R}$, then $C_1(J)$ denotes the class of all functions with a continuous derivative on $J$; $C_1$ means $C_1(\mathbb{R})$.

**2. Lemma.** *Let $a, b \in \mathbb{R}$, $a < b$, $J = [a, b]$. Let $f$ be integrable on $J$ and let $\varepsilon \in \mathbb{R}^+$. Then there is a $g \in C_1$ such that $g = 0$ on $\mathbb{R} \setminus J$, $0 \le g \le 1$ on $J$ and $|\int_J f - \int_J fg| < \varepsilon$.*

Proof. There is a $\delta \in (0, |J|/2)$ such that $|\int_x^y f| < \varepsilon/2$, whenever $a \le x < y \le b$ and $y - x \le \delta$. Set $\alpha = a + \delta$, $\beta = b - \delta$. There is a $g \in C_1$ such that $g = 0$

on $\mathbb{R}\setminus J$, $g=1$ on $(\alpha,\beta)$, $g$ is monotone on $(a,\alpha)$ and on $(\beta,b)$. By the Second Mean Value Theorem (see, e.g., [S], p. 246, Theorem (2.6)) there are $\xi\in[a,\alpha]$ and $\eta\in[\beta,b]$ such that $\int_a^\alpha f(1-g)=\int_a^\xi f$, $\int_\beta^b f(1-g)=\int_\eta^b f$. We see that $g$ satisfies our requirements.

**3. Lemma.** *Let $a,b,J,f$ be as before and let $Q$ be a number less than $\int_J |f|$. Then there is a $g\in C_1$ such that $g=0$ on $\mathbb{R}\setminus J$, $|g|\le 1$ on $J$ and $\int_J fg>Q$.*

Proof. Let $V$ be the variation of an indefinite integral of $f$ on $J$. Then $\int_J |f| = V$. (This is well-known, if $\int_J |f|$ or $V$ is finite; hence it holds even if $\int_J |f|=\infty$.) It follows that there is an $\varepsilon\in\mathbb{R}^+$ and $x_0,\dots,x_n\in J$ such that $a=x_0<x_1<\dots<x_n=b$ and that, setting $J_k=[x_{k-1},x_k]$ and $A_k=\int_{J_k} f$, we have $\sum_{k=1}^n |A_k|>Q+\varepsilon$. By 2 there are functions $g_k\in C_1$ such that $g_k=0$ on $\mathbb{R}\setminus J_k$, $0\le g_k\le 1$ on $J_k$ and $|A_k-\int_{J_k} fg_k|<\varepsilon/n$. It is easy to see that the function $g=\sum_{k=1}^n g_k \operatorname{sgn} A_k$ satisfies our requirements.

**4. Convention.** Symbols like $\limsup f(x)$, $f(x)\to 0$ etc. will refer to the case $x\to 0+$, unless something else is obvious from the context.

**5. Lemma.** *Let $f$ be a function such that $\frac{1}{x}\int_0^x fg\to 0$ for each $g\in C_1(\mathbb{R}^+)$ with $g(0+)=0$. Then*

$$\limsup \frac{1}{x}\int_0^x |f|<\infty. \tag{1}$$

Proof. It is easy to see that $f$ is measurable on $(0,\delta)$ for some $\delta\in\mathbb{R}^+$. Now suppose that (1) does not hold. Then there are $x_n,y_n\in\mathbb{R}$ such that $0<x_n<y_n<x_{n-1}$, $y_n\to 0$ and that, setting $J_n=[x_n,y_n]$, we have $\int_{J_n}|f|>ny_n$ $(n=1,2,\dots)$. By 3 there are $g_n\in C_1$ such that $g_n=0$ on $\mathbb{R}\setminus J_n$, $|g_n|\le 1$ on $J_n$ nad $\int_{J_n} fg_n>ny_n$. Set $g=\sum_{n=1}^\infty g_n/n$ on $\mathbb{R}^+$. Then $g\in C_1(\mathbb{R}^+)$ and $g(0+)=0$. By assumption $\frac{1}{x}\int_0^x fg\to 0$. It follows that there are $\alpha_n,\beta_n\in\mathbb{R}$ such that $|\alpha_n|+|\beta_n|\to 0$ $(n\to\infty)$, $\int_0^{x_n} fg=\alpha_n x_n$, $\int_0^{y_n} fg=\beta_n y_n$. Then $\int_{J_n} fg\le y_n(|\alpha_n|+|\beta_n|)$. However, $\int_{J_n} fg=\frac{1}{n}\int_{J_n} fg_n>y_n$ for each $n$ which is a contradiction.

**6. Proposition.** *Let $f$ be a measurable function on $I$. Then the following three conditions are equivalent:*

(i) $\frac{1}{x}\int_0^x fg\to 0$ *for each* $g\in C_1(\mathbb{R}^+)$ *with* $g(0+)=0$;

(ii) $\limsup\frac{1}{x}\int_0^x |f|<\infty$;

(iii) $\frac{1}{x}\int_0^x fg\to 0$ *for each measurable function* $g$ *on* $I$ *with* $g(0+)=0$.

Proof. The implication (i) $\Rightarrow$ (ii) was proved in 5. The proof of the implication (ii) $\Rightarrow$ (iii) is left to the reader. The implication (iii) $\Rightarrow$ (i) is obvious.

**7. Theorem.** *Let $f \in D$. Then the following two conditions are equivalent:*

(i) *$fg \in D$ for each function $g$ continuous on $I$;*

(ii) $\limsup \frac{1}{y-x} \int_x^y |f| < \infty$ *$(y \to x, y \in I)$ for each $x \in I$.*

(This follows easily from 6.)

**Remark.** It follows from 7 that the product of a nonnegative derivative with a continuous function is always a derivative. However, it is easy to prove this simple result directly.

On the other hand it is worth mentioning that the product of a Lebesgue integrable derivative with a continuous function need not be a derivative. (A Lebesgue integrable derivative need not be the difference of two nonnegative derivatives.) To see this it suffices to take $f(x) = x^{-1/2}\sin(1/x)$, $g(x) = x^{1/2}\sin(1/x)$ $(x \in (0,1]$, $f(0) = g(0) = 0$.

**8. Lemma.** *Let $\delta, A \in \mathbb{R}^+$. Let $f$ be a nonnegative measurable function on $(0,\delta)$ such that $\int_0^\delta f > \delta A$. Then there is an $x \in (0, \delta/2]$ such that $\int_x^{2x} f > xA$.*

Proof. We may choose $x = \delta/2^n$ for some $n \in \{1, 2, \ldots\}$.

**9. Lemma.** *Let $f$ be a measurable function on $I$. Suppose that*

$$\frac{1}{x}\int_{S\cap(0,x)} f \to 0 \tag{2}$$

*for each measurable set $S \subset I$ with $d(S,0) = 0$. Then*

$$\frac{1}{x}\int_{S\cap(0,x)} |f| \to 0 \tag{3}$$

*for every such $S$ and*

$$\limsup \frac{1}{x}\int_0^x |f| < \infty. \tag{4}$$

Proof. Let $S$ be as above. Set $g = f \vee 0$, $T = S \cap \{f > 0\}$. Since $d(T,0) = 0$, $\int_{S\cap(0,x)} g = \int_{T\cap(0,x)} f$ and $|f| = 2g - f$, we have (3).

Now suppose that (4) does not hold. Using 8 we find $x_n \in I$ such that $0 < x_n < x_{n-1}/2$ and $\int_{x_n}^{2x_n} |f| > nx_n$ $(n = 1, 2, \ldots)$. Set $x_{nk} = x_n(1 + k/n)$, $J_{nk} =$

$[x_{n,k-1}, x_{nk}]$. For each $n$ there is a $k \in \{1, \dots, n\}$ such that $\int_{J_{nk}} |f| > x_n$. Let $L_n = J_{nk}$, $S = \bigcup_{n=1}^{\infty} L_n$. It is easy to see that $d(S, 0) = 0$. For $x = 2x_n$ we have $\int_{S\cap(0,x)} |f| \geqq \int_{L_n} |f| > x_n = x/2$ so that (3) does not hold. This contradiction proves (4).

**10. Lemma.** *Let $\varepsilon, \delta \in (0,1)$ and let $f$ be as in 9. Suppose, moreover, that $\int_0^\delta |f| < \infty$. For each $c \in \mathbb{R}$ and each $x \in (0,1)$ set $M(c,x) = \{t \in (0,x); |f(t)| \geqq c\}$. Then there is a $c \in \mathbb{R}$ such that $\int_{M(c,x)} |f| \leq \varepsilon x$ for each $x \in (0,\delta)$.*

Proof. Suppose that such a $c$ does not exist. By 9 there is a $K \in \mathbb{R}^+$ such that $\int_0^x |f| < Kx$ for each $x \in (0,\delta)$. Set $c_0 = 0$, $x_0 = \delta$. We construct by induction numbers $x_n, y_n, c_n$ as follows: Let $x_{n-1} \in (0,\delta]$ and let $c_{n-1} \in [0,\infty)$. There is a $c_n \in (c_{n-1}+1, \infty)$ such that $\int_{M(c_n,\delta)} |f| < \varepsilon x_{n-1}/2$. By assumption there is a $y_n \in (0,\delta)$ such that $\int_{M(c_n,y_n)} |f| > \varepsilon y_n$. Clearly $y_n < \varepsilon^{-1} \int_{M(c_n,\delta)} |f| < x_{n-1}/2$. Now we find an $x_n \in (0, y_n)$ such that $\int_{S_n} |f| > \varepsilon y_n$, where $S_n = M(c_n, \delta) \cap (x_n, y_n)$. Set $S = \bigcup_{n=1}^{\infty} S_n$. Let $x_n < x \leq x_{n-1}$. Since $S \cap (0,x) \subset \bigcup_{k=n}^{\infty} S_k$ and $|f| \geqq c_k \geqq k$ on $S_k$, we have $|S \cap (0,x)| \leq \int_0^x |f|/n < Kx/n$. Thus $d(S,0) = 0$. However, $\int_{S\cap(0,y_n)} |f| \geqq \int_{S_n} |f| > \varepsilon y_n$ which contradicts (3).

**11. Proposition.** *Let $f$ be a measurable function on $I$. Then the following four conditions are equivalent:*

(i) $\frac{1}{x}\int_0^x fg \to 0$ *for each function $g$ bounded and continuous on $(0,1]$ with* $\lim \operatorname{ap} g(x) = 0$;

(ii) $\frac{1}{x}\int_{S\cap(0,x)} f \to 0$ *for each measurable set $S \subset I$ with $d(S,0) = 0$;*

(iii) *there is a monotone function $\varphi$ on $[0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t)/t \to \infty$ $(t \to \infty)$ and* $\limsup \frac{1}{x}\int_0^x \varphi \circ |f| < \infty$;

(iv) $\frac{1}{x}\int_0^x fg \to 0$ *for each function $g$ bounded and measurable on $I$ with* $\lim \operatorname{ap} g(x) = 0$.

Proof. Suppose that (i) holds and let $S$ be as in (ii). It follows from 5 that $\int_0^\delta |f| < \infty$ for some $\delta \in (0,1)$. Let $h$ be the characteristic function of $S$. It is easy to construct a function $g$ continuous on $(0,1]$ such that $0 \leq g \leq 1$ and that

$$\frac{1}{x}\int_0^x (1+|f|)|g-h| \to 0. \tag{5}$$

Since $\frac{1}{x}\int_0^x h \to 0$, we have also $\frac{1}{x}\int_0^x g \to 0$ whence $\lim \operatorname{ap} g(x) = 0$. By assumption $\frac{1}{x}\int_0^x fg \to 0$ so that, by (5), $\frac{1}{x}\int_0^x fh \to 0$. This proves (ii).

Suppose that (ii) holds. By 9 there is a $\delta \in (0,1)$ such that $\int_0^\delta |f| < \infty$. Choose numbers $\varepsilon_n \in (0,1)$ such that $\sum_{n=1}^\infty n\varepsilon_n \leq 1$.

Set $c_0 = 0$. According to 10 there are $c_n \in \mathbb{R}$ such that $c_n > c_{n-1}+1$ and that $\int_{M(c_n,x)} |f| \leq \varepsilon_n x$ for each $x \in (0,\delta)$ $(n = 1,2,\ldots)$. For $t \in [c_n, c_{n+1})$ set $\varphi(t) = nt$ $(n = 0,1,\ldots)$. Now let $x \in (0,\delta)$. Define $A_n = \{t \in (0,x);\ c_n \leq |f(t)| < c_{n+1}\}$. Clearly $A_n \subset M(c_n,x)$, $\varphi \circ |f| = n|f|$ on $A_n$ and $(0,x) = \bigcup_{n=0}^\infty A_n$. Hence $\int_0^x \varphi \circ |f| \leq \sum_{n=0}^\infty n \int_{M(c_n,x)} |f| \leq x \sum_{n=1}^\infty n\varepsilon_n \leq x$. This proves (iii).

Suppose that (iii) holds and let $g$ be as in (iv). There is a $\delta \in (0,1)$ and $A, B \in \mathbb{R}^+$ such that $|g| < A$ on $I$ and that $\int_0^x \varphi \circ |f| < Bx$ for each $x \in (0,\delta)$. Let $\varepsilon \in \mathbb{R}^+$ and let $Q = AB/\varepsilon$. There is a $K \in \mathbb{R}^+$ such that $\varphi(v) > Qv$ for each $v \in (K,\infty)$. If $|f(t)| > K$, then $|f(t)| < \varphi(|f(t)|)/Q$. Thus $|\int_0^x fg| \leq K\int_0^x |g| + \frac{A}{Q}\int_0^x \varphi \circ |f|$ whence $\limsup |\frac{1}{x}\int_0^x fg| \leq AB/Q = \varepsilon$. This proves (iv).

It is obvious that (i) follows from (iv). This completes the proof.

**12. Theorem.** *Let $f \in D$. Then the following three conditions are equivalent:*

(i) *For each $x \in I$ and each measurable set $S \subset I$ with $d(S,x) = 0$ we have $\frac{1}{h}\int_{S\cap(x-h,x+h)} f \to 0$ $(h \to 0+)$;*

(ii) *for each $x \in I$ there is a monotone function $\varphi$ on $[0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t)/t \to \infty (t \to \infty)$ and*

$$\limsup \frac{1}{y-x}\int_x^y \varphi \circ |f| < \infty \ (y \to x, y \in I);$$

(iii) *$fg \in D$ for each bounded Lebesgue function $g$.*

(This follows easily from 11.)

**13. Remark.** Theorem 12 characterizes multipliers of bounded Lebesgue functions. Now we would like to get an idea about the "size" of this system; let us denote it by $M$. It is easy to prove that the product of a Lebesgue function with a bounded derivative is always a derivative; thus all Lebesgue functions are in $M$. From 12 we see that $M$ contains, for example, every derivative $f$ such that $\limsup \frac{1}{y-x}\int_x^y f^2 < \infty$ $(y \to x, y \in I)$ for each $x \in I$. Proposition 5.8 in [MW] says that an approximately continuous function is in $M$ if and only if it is a Lebesgue function.

Now let $E$ be the vector space generated by nonnegative derivatives. (It is easy to see that a derivative $f$ is in $E$ if and only if $|f| \leq g$ for some $g \in D$.) It has already been mentioned that each nonnegative derivative (and so each element of $E$) is a multiplier of continuous functions. It may interest the reader that we have neither $E \subset M$ nor $M \subset E$. To show that $E \not\subset M$ it suffices to construct functions $f$ and $g$ such that $f \geqq 0$ and $0 \le g \le 1$ on $I$, $f$ and $g$ are continuous on $(0,1]$, $f \in D$, $f(0) = 1$, $g(0) = 0 = \lim \text{ap}\, g(x)$ and that $f(x) = 0$, whenever $x \in (0,1]$ and $g(x) < 1$. Then $fg = f$ on $(0,1]$ while $(fg)(0) = 0$ so that $fg \notin D$, $f \in E \setminus M$. To show that $M \not\subset E$ is not so easy. We shall construct an $f \in M \setminus E$ in 15. First we prove a simple lemma.

**14. Lemma.** *Let $f, g$ be measurable functions on $I$, $|f| \le g$. Let $Q \in \mathbb{R}$, $\frac{1}{x}\int_0^x g \to Q$. Let $c \in (1,\infty)$. Then*

$$\limsup \frac{1}{(c-1)x}\int_x^{cx} |f| \le Q.$$

Proof. Set $G(x) = \int_0^x g$. Then $\frac{1}{x}\int_x^{cx} |f| \le \frac{1}{x}(G(cx) - G(x)) \to (c-1)Q$.

**15. Example.** There is a function $f \in D \setminus E$ such that $f$ is continuous on $(0,1]$ and $\limsup \frac{1}{x}\int_0^x f^2 < \infty$ (hence $f \in M$).

Proof. Let $F$ be a function continuous and decreasing on $(0,1]$ such that $F(0+) = \infty$, $F(1) = 1$ and $\int_0^1 F^2 < \infty$. Set $A = \int_0^1 F$. Let $n$ be a positive even number. Let $x_k$ be numbers such that $0 = x_0 < x_1 < \cdots < x_n = 1$, $\int_{x_{k-1}}^{x_k} F = A/n$. Let $y_k, z_k$ be numbers such that $\int_{x_{k-1}}^{y_k} F = \int_{z_k}^{x_k} F = 1/n^2$. Since $A > 1$ and $n \geqq 2$, we have $\frac{2}{n^2} < \frac{A}{n}$ so that $x_{k-1} < y_k < z_k < x_k$. Let $g_k$ be a function continuous on $[x_{k-1}, x_k]$ such that $0 \le g_k \le F$ there, $g_k(x_{k-1} = g_k(x_k) = 0$, $g_k = F$ on $[y_k, z_k]$, $\int_{x_{k-1}}^{y_k} g_k = \int_{z_k}^{x_k} g_k = 1/2n^2$. Now we define a function $F_n$ on $I$ setting $F_n = g_k(-1)^{k-1}$ on $[x_{k-1}, x_k]$ $(k = 1, \ldots, n)$. It is easy to see that $F_n$ is continuous on $I$, $|F_n| \le F$, $\int_{x_{k-2}}^{x_k} F_n = 0$ $(k = 2, \ldots, n)$ and $0 \le \int_0^x F_n < A/n$ for each $x \in I$. Let $V_n = \{x \in I; |F_n(x)| < F(x)\}$, $W_n = \bigcup_{k=1}^n ((x_{k-1}, y_k) \cup (z_k, x_k))$. Since $V_n \subset W_n$, we have $\int_{V_n} F \le 2n^{-2} \cdot n = 2/n$.

Now set $z_k = 2^{-k}$ $(k = 1, 2, \ldots)$. Define a function $f$ on $I$ setting $f(0) = 0$ and $f(x) = F_{2k}((x - z_k)/z_k)$ for $x \in (z_k, 2z_k]$. For any such $x$ we have $0 \le \int_{z_k}^x f \le z_k A/2k$. Clearly $\int_{z_k}^{2z_k} f^2 \le z_k \int_I F^2$ and $\int_{z_k}^{2z_k} f = 0$. Hence $\limsup \frac{1}{x}\int_0^x f^2 < \infty$ and $\frac{1}{x}\int_0^x f \to 0$. Since $f$ is continuous on $(0,1]$, we have, by 12, $f \in M$.

Let $B \in (1,\infty)$. There is a $b \in (0,1)$ such that $F(b) = B$. Set $v_k = z_k(1+b)$. (Hence $v_k - z_k = z_k b$.) Then $\int_{z_k}^{v_k} |f| = z_k \int_0^b |F_{2k}|$. Define $S_k = (0,b) \setminus V_{2k}$. Obviously $|S_k| \geqq b - |V_{2k}|$; since $F \geqq 1$, we have $|V_{2k}| \le \int_{V_{2k}} F \le 1/k$ so that $|S_k| \geqq b - 1/k$. Further $\int_0^b |F_{2k}| \geqq \int_{S_k} |F_{2k}| = \int_{S_k} F \geqq |S_k| F(b) \geqq (b - 1/k)B$, $\int_{z_k}^{v_k} |f| \geqq z_k(b - 1/k)B = (v_k - z_k)(1 - (kb)^{-1})B$, $\liminf \frac{1}{v_k - z_k}\int_{z_k}^{v_k} |f| \geqq B (k \to \infty)$. It follows from 14 that there is no $g \in D$ with $|f| \le g$. Hence $f \notin E$.

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