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A Note on Real Cliquish Functions

In [9] it is proved that every cliquish function $f : \mathbb{R}^m \to \mathbb{R}$ is the sum of six quasicontinuous functions. By [5] every cliquish function $f : \mathbb{R} \to \mathbb{R}$ is the sum of four quasicontinuous functions. In this paper we show that every cliquish function $f : \mathbb{R}^m \to \mathbb{R}$ is the sum of two simply continuous functions each of which is the sum of two quasicontinuous functions.

In what follows X denote a topological space. For a subset A of a topological space denote $C\ell A$ and $\operatorname{Int} A$ the closure and the interior of A, respectively. The letters $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively.

We recall that a function $f : X \to \mathbb{R}$ is cliquish at a point $x \in X$ (see [6]) if for each $\varepsilon > 0$ and each neighborhood U of x there is a nonempty open set $G \subset U$ such that $|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$. A function $f : X \to \mathbb{R}$ is said to be cliquish if it is cliquish at each point $x \in X$.

A function $f : X \to \mathbb{R}$ is simply continuous (see [1]) if for each open set V in $\mathbb{R}$, the set $f^{-1}(V)$ is the union of an open set and a nowhere dense set in X.

A function $f : X \to \mathbb{R}$ is quasicontinuous at a point $x \in X$ (see [6]) if for each neighborhood U of x and each neighborhood V of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by Q_f the set of all points at which f is quasicontinuous. If $Q_f = X$, then f is said to be quasicontinuous.

It is easy to see that every quasicontinuous function is simply continuous and cliquish. In [7] it is shown that if X is a Baire space, then every simply continuous function $f : X \to \mathbb{R}$ is cliquish. Example 1 in [3] shows that the assumption "X is a Baire space" cannot be omitted.

If $f : X \to \mathbb{R}$ is cliquish, then $X \setminus C_f$ (where C_f is the set of all continuity points of f) is of the first category in X (see [6]). If X is a Baire space, then $f : X \to \mathbb{R}$ is cliquish if and only if C_f is dense in X (see [4]).

We recall that a π-base for X is a family $\mathcal{A}$ of open subsets of X such that every nonempty open subset of X contains some nonempty $A \in \mathcal{A}$ (see [8]).

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Lemma 1 *Let X be a topological space such that the family of all open connected sets is a π-base for X. Let $h : X \to \mathbb{R}$ be a cliquish function such that $h^{-1}(0)$ is dense in X. Let $g : X \to \mathbb{R}$ be a continuous function which is constant on no nonempty open subset of X. Then $f = g + h$ is simply continuous.*

Proof. Let $u, v \in \mathbb{R}$ and $\varepsilon > 0$ be such that $u + \varepsilon < v - \varepsilon$. Put

$$A_{u,v,\varepsilon} = \{x \in g^{-1}((u+\varepsilon, v-\varepsilon)) : f(x) \notin (u,v)\}.$$

Then $A_{u,v,\varepsilon} = \{x \in A_{u,v,\varepsilon} : h(x) \geq 0\} \cup \{x \in A_{u,v,\varepsilon} : h(x) \leq 0\} \subset \{x \in g^{-1}((-\infty, v-\varepsilon)) : g(x) + |h(x)| \geq v\} \cup \{x \in g^{-1}((u+\varepsilon, \infty)) : g(x) - |h(x)| \leq u\} \subset \{x \in X : |h(x)| > \varepsilon\}$. Hence $A_{u,v,\varepsilon}$ is nowhere dense in X.

Let $V \subset \mathbb{R}$ be an open set. We shall show that

$$g^{-1}(V) \subset C\ell \operatorname{Int} f^{-1}(V). \tag{1}$$

Let $x \in g^{-1}(V)$. Let $u, v \in \mathbb{R}$ and $\varepsilon > 0$ be such that $g(x) \in (u+\varepsilon, v-\varepsilon) \subset (u,v) \subset V$. Let U be a neighborhood of x such that $g(U) \subset (u+\varepsilon, v-\varepsilon)$. Since $A_{u,v,\varepsilon}$ is nowhere dense in X, there is a nonempty open set $G \subset U$ such that $G \cap A_{u,v,\varepsilon} = \emptyset$. Since $G \subset f^{-1}((u,v)) \subset f^{-1}(V)$, we have $G \subset \operatorname{Int} f^{-1}(V)$. Thus $x \in C\ell \operatorname{Int} f^{-1}(V)$.

Put $W = f^{-1}(V) \setminus \operatorname{Int} f^{-1}(V)$. We shall show that W is nowhere dense in X. Let $J \subset X$ be a nonempty open connected set. Put $H = J \cap g^{-1}(V)$. We distinguish two cases.

a) Suppose that $H \neq \emptyset$. By (1) we obtain $W \cap g^{-1}(V) \subset C\ell \operatorname{Int} f^{-1}(V) - \operatorname{Int} f^{-1}(V)$. Hence there is a nonempty open set $E \subset H$ such that

$$\emptyset = E \cap W \cap g^{-1}(V) = E \cap W.$$

b) Suppose that $H = \emptyset$. Let $u, v \in \mathbb{R}$ and $\varepsilon > 0$ be such that $u + \varepsilon < v - \varepsilon$ and $(u,v) \subset g(J)$. Then there is a nonempty open set $T \subset J$ such that $g(T) \subset (u+\varepsilon, v-\varepsilon)$. Since $A_{u,v,\varepsilon}$ is nowhere dense in X, there is a nonempty open set $F \subset T$ such that $F \cap A_{u,v,\varepsilon} = \emptyset$. We shall show that $F \cap W = \emptyset$. Let $x \in F$. Then $x \in g^{-1}((u+\varepsilon, v-\varepsilon))$, which yields $f(x) \in (u,v) \subset g(J)$. Therefore there is $z \in J$ such that $f(x) = g(z)$. Since $H = \emptyset$, we obtain $f(x) \notin V$. Therefore $x \notin W$.

The following example shows that the assumption of the continuity of g in Lemma 1 cannot be replaced by the assumption of the quasicontinuity of g.

Example 1 Let $Q = \{q_1, q_2, q_3, \ldots\}$. Define $g, h : \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \sum_{n:q_n<x} 2^{-n} \quad \text{for each } x \in \mathbb{R}$$

$$h(x) = \begin{cases} 2^{-n-1}, & \text{for } x = q_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then g is a quasicontinuous function which is constant on no nonempty open subset of $\mathbb{R}, h$ is a cliquish function such that $h^{-1}(0)$ is dense in $\mathbb{R}$. However $f = g + h$ is not simply continuous.

Remark 1 The sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin x + h(x)$, where h is the function from Example 1, shows that the class of all simply continuous functions is not closed with respect to uniform convergence.

Perhaps the following Lemma is known but we are not able to give any references.

Lemma 2 *Let X be a second countable T_3-space without isolated points. Then there is a continuous function $g : X \to \mathbb{R}$ which is constant on no nonempty open subset of X.*

Proof. Let $\mathcal{B} = \{B_1, B_2, B_3, \ldots\}$ be a countable base for X. For each $n \in \mathbb{N}$ choose $y_n, z_n \in B_n$ such that $y_n \neq z_n$. Let $g_1 : X \to [0,1]$ be a continuous function such that $g_1(y_1) = 0$ and $g_1(z_1) = 1$. Suppose $g_1, \ldots, g_k$ have been constructed. Let $h_{k+1} : X \to [0,1]$ be a continuous function such that $h_{k+1}(y_{k+1}) = 0$ and $h_{k+1}(z_{k+1}) = 1$. Put

$$g_{k+1} = \begin{cases} h_{k+1}, & \text{if } \sum_{i=1}^{k} 4^{-i} g_i(y_{k+1}) \leq \sum_{i=1}^{k} 4^{-1} g_i(z_{k+1}), \\ 1 - h_{k+1}, & \text{otherwise.} \end{cases}$$

Put $f = \sum_{i=1}^{\infty} 4^{-i} g_i$. Then f is a continuous function which is constant on no nonempty open subset of X.

Theorem 1 *Let X be a Baire second countable T_3-space such that the family of all open connected sets is a π-base for X. Then every cliquish function $f : X \to \mathbb{R}$ is the sum of two simply continuous functions.*

Proof. Denote by D the set of all isolated points of X. Put $B = X \setminus C\ell\, D$ Then by Lemma 2 there is a continuous function $h : B \to \mathbb{R}$ which is constant on no nonempty open subset of X. Denote by A the set of all points in X at which

f is not locally bounded. Since C_f is dense in X and A is closed, A is nowhere dense in X. Define $G : B \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \limsup_{u \to x, u \in C_f}(f(u) - h(u)) & \text{if } x \in B \setminus A \\ f(x) - h(x) & \text{otherwise.} \end{cases}$$

Evidently

$$g(x) = f(x) - h(x) \text{ for each } x \in C_f \cap B. \tag{2}$$

Let $x \in B \setminus A$. Let $U \subset B$ be a neighborhood of x and $\varepsilon > 0$. Then there is $u \in C_f \cap U$ such that $|f(u) - h(u) - g(x)| < \frac{\varepsilon}{2}$. Furthermore there is an open neighborhood $G \subset U$ of u such that $|g(u) - g(y)| < \frac{\varepsilon}{2}$ for each $y \in G$. Therefore for each $y \in G$ we have $|g(x) - g(y)| \leq |g(x) - f(u) + h(u)| + |f(u) - h(u) - g(u)| + |g(u) - g(y)| < \varepsilon$. This shows that

$$B \setminus A \subset Q_g. \tag{3}$$

Define $k : B \to \mathbb{R}$ as follows

$$k(x) = f(x) - h(x) - g(x) \text{ for each } x \in B.$$

Since g is cliquish on B, the function k is cliquish on B. From (2) we obtain that $k^{-1}(0)$ is dense in B. According to Lemma 1 we have that $k + h$ is simply continuous. Define $f_1, f_2 : X \to \mathbb{R}$ as follows

$$f_1(x) = \begin{cases} g(x), & \text{if } x \in B, \\ f(x), & \text{otherwise;} \end{cases}$$

$$f_2(x) = \begin{cases} k(x) + h(x), & \text{if } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently $f_1 + f_2 = f$. We shall show that f_1 and f_2 are simply continuous.

Since $D \subset Q_{f_1}$, by (3) we have $X \setminus Q_{f_1} \subset A \cup (\mathrm{C}\ell\, D \setminus D)$. Thus $X \setminus Q_{f_1}$ is nowhere dense in X. Let V be an open set in $\mathbb{R}$. According to Remark 1 and Lemma 1 in [2] the set $Q_{f_1} \cap (f_1^{-1}(V) \setminus \mathrm{Int}\, f_1^{-1}(V))$ is nowhere dense in X. Therefore $f_1^{-1}(V) \setminus \mathrm{Int}\, f_1^{-1}(V) \subset ((f_1^{-1}(V) \setminus \mathrm{Int}\, f_1^{-1}(V)) \cap Q_{f_1}) \cup (X \setminus Q_{f_1})$ is nowhere dense in X. Thus f_1 is simply continuous.

Finally $f_2^{-1}(V) = (f_2^{-1}(V) \cap B) \cup (f_2^{-1}(V) \cap \mathrm{C}\ell\, D)$ is the union of two sets each of which is the union of an open set and a nowhere dense set in X. Thus f_2 is simply continuous.

The following example shows that the assumptions in Theorem 1 cannot be omitted.

Example 2 Let $X = \mathbb{R}$ with the cofinite topology. Evidently X is Baire and locally connected. Define $g : X \longrightarrow \mathbb{R}$ as follows

$$g(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since for each simply continuous function $f : X \longrightarrow \mathbb{R}$ the set $f(X)$ is finite, the cliquish function g is not a sum of finitely many simply continuous functions.

Lemma 3 *Let X be a Baire separable locally connected metric space. Let $f : X \longrightarrow \mathbb{R}$ be such that the set $X \setminus Q_f$ is nowhere dense in X. Then f is the sum of two quasicontinuous functions.*

Proof. Let $\mathcal{B}$ be a countable base for X. Put $\mathcal{A} = \{B \in \mathcal{B} : C\ell B \subset \mathrm{Int}\, Q_f\}$. Then $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$. Let $W = \{w_1, w_2, w_3, \ldots\}$ be a countable dense subset in $X \setminus \mathrm{Int}\, Q_f$. Let $i \in \mathbb{N}$. Since $X \setminus \bigcup_{j=1}^{i} C\ell A_j$ is an open neighborhood of the point w_i, there is a sequence $(v_j^i)_j$ of points such that $v_j^i \in (\mathrm{Int}\, Q_f \cap C_f) \setminus \bigcup_{k=1}^{i} C\ell A_k$ and $(v_j^i)_j$ converges to w_i. Put $E = \{v_j^i : i, j \in \mathbb{N}\}$. It is not difficult to verify that E is discrete. Define functions $f_1, f_2 : X \longrightarrow \mathbb{R}$ as follows

$$\begin{aligned} f_1(x) &= \begin{cases} \frac{1}{\mathrm{dist}(x, C\ell E)} \sin\left(\frac{1}{\mathrm{dist}(x, C\ell E)}\right), & \text{if } x \in X \setminus C\ell E, \\ 0, & \text{if } x \in C\ell E; \end{cases} \\ f_2(x) &= f(x) - f_1(x). \end{aligned}$$

We shall show that f_2 is quasicontinuous. Similarly we can show that f_1 is quasicontinuous.

Evidently f_2 is quasicontinuous on $X \setminus C\ell E$.

Let $x \in C\ell E$. We may assume that x is not an isolated point of X. Let U be an open connected neighborhood of x and $\varepsilon > 0$. Choose $y \in U \cap E$. Then y is not an isolated point of X. Since $y \in C_f$, there is an open neighborhood V of y such that

$$|f(t) - f(y)| < \frac{\varepsilon}{2} \text{ for each } t \in V.$$

Let $W \subset U \cap V$ be a neighborhood of y such that $W \cap E = \{y\}$. Since f_1 assumes any real value on W, we obtain

$$\exists z \in W, z \neq y : f_1(z) = f(y) - f(x).$$

Since f_1 is continuous at z, there is an open neighborhood $G \subset W$ of z such that

$$|f_1(t) - f_1(z)| < \frac{\varepsilon}{2} \text{ for each } t \in G.$$

Let $t \in G$. Then we have

$$|f_2(x) - f_2(t)| = |f(x) - f(t) + f_1(t)| \leq$$
$$|f(x) - f(y) + f_1(z)| + |f(y) - f(t)| + |f_1(t) - f_1(z)| < \varepsilon.$$

Thus f_2 is quasicontinuous at x. Evidently $f = f_1 + f_2$.

Proposition 1 (See [9; Theorem 3].) *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a cliquish function such that the set $f^{-1}(0)$ is dense in $\mathbb{R}^m$. Then f is the sum of two quasicontinuous functions.*

Theorem 2 *Every cliquish function $f : \mathbb{R}^m \to \mathbb{R}$ is the sum of two simply continuous functions each of which is the sum of two quasicontinuous functions.*

Proof. Let $h : \mathbb{R}^m \to \mathbb{R}$ be a continuous function which is constant on no nonempty open set. From the proof of Theorem 1 it follows that there are simply continuous functions $f_1, f_2 : \mathbb{R}^m \to \mathbb{R}$ such that $f = f_1 + f_2, \mathbb{R}^m \setminus Q_{f_1}$ is nowhere dense in $\mathbb{R}^m, f_2 - h$ is cliquish and $(f_2 - h)^{-1}(0)$ is dense in $\mathbb{R}^m$. By Lemma 3 there are quasicontinuous functions $g_1, g_2 : \mathbb{R}^m \to \mathbb{R}$ such that $f_1 = g_1 + g_2$. According to Proposition 1 there are quasicontinuous functions $g_3, g_5 : \mathbb{R}^m \to \mathbb{R}$ such that $f_2 - h = g_3 + g_5$. Put $g_4 = g_5 + h$. Then g_4 is quasicontinuous and $f_2 = g_3 + g_4$.

References

[1] Biswas, N., *On some mappings in topological spaces*, Bull. Calcutta Math. Soc. 61 (1969), 127-135.

[2] Borsík, J. and Doboš, J., *On simple continuity points*, Real Analysis Exchange 16 (1990-91), 552-558.

[3] Doboš, J., *Simple continuity and cliquishness*, Časopis pěst. mat. 112 (1987), 355-358.

[4] Doboš, J. and Šalát, T., *Cliquish functions, Riemann integrable functions and quasi-uniform convergence*, Acta Math. Univ. Comen. 40-41 (1982), 219-223.

[5] Grande, Z., *Sur les fonctions cliquish*, Časopis pěst. mat. 110 (1985), 225-236.

[6] Marcus, S., *Sur les fonctions quasicontinues au sens de S. Kempisty*, Colloq. Math. 8 (1961), 47-53.

[7] Neubrunnová, A., *On certain generalizations of the notion of continuity*, Mat. časopis 23 (1973), 374-380.

[8] Oxtoby, J.C., *Cartesian product of Baire spaces*, Fund. math. 49 (1961), 156-170.

[9] Stronska, E., *L'espace linéaire des fonctions cliquées sur R^n est généré par les fonctions quasi-continues*, Math. Slovaca 39 (1989), 155-164.