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On the Darboux Property of Restricted Functions

Let $\mathbb{R}$ denote the set of reals. If $A \subset \mathbb{R}$ is a nonempty set, then we say that a function $f : A \to \mathbb{R}$ has the Darboux property whenever $f(I \cap A)$ is a connected set for every interval $I \subset \mathbb{R}$. Denote by $D(A)$ $(A \neq \emptyset)$ the set of all functions $f : A \to \mathbb{R}$ having the Darboux property. Let $C(A)$ denote the family of all continuous functions $f : A \to \mathbb{R}$ and let p be the uniform metric defined by the following formula

$$p(f,g) = \min(1, \sup_{x \in A} |f(x) - g(x)|).$$

Theorem 1 *If a set $A \subset \mathbb{R}$ containing more than one point, is not an interval, then the set $C(A) \setminus D(A)$ has a nonempty interior (in the metric p).*

Proof. There is a point $a \in \mathbb{R} \setminus A$ such that $(-\infty, a) \cap A \neq \emptyset$ and $(a, \infty) \cap A \neq \emptyset$. Let $b = \sup(A \cap (-\infty, a))$ and $c = \inf(A \cap (a, \infty))$. There is a continuous function $f : A \to \mathbb{R}$ such that:

$$\lim_{x \to b-} f(x) = 0;$$

$$\lim_{x \to c+} f(x) = 1.$$

For every function $g \in C(A)$ with $p(f,g) < 1/2$ there is $r > 0$ such that $g(x) < 1/2$ for every $x \in A \cap (b - r, b]$ and $g(y) > 1/2$ for every $y \in A \cap [c, c + r)$. Since $1/2 \notin g(A \cap (b - r, c + r))$, g does not have the Darboux property. This completes the proof.

Theorem 2 *If there exist points $a, b \in A$ such that $a < b$ and the intersection $[a,b] \cap A$ has cardinality smaller than the continuum, then the set $C(A) \cap D(A)$ is nowhere dense in $C(A)$.*

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Proof. Let $H(A)$ be the set $\{f : A \to \mathbb{R};\ f$ constant on $[a,b] \cap A\}$. Obviously, $H(A)$ is uniformly closed. Let $f \in C(A) \cap H(A)$ be a fixed function and let $r > 0$ be a number < 1. Define

$$g(x) = \begin{cases} f(x) & \text{for } x \in A \cap (-\infty, a] \\ f(x) + r/2 & \text{for } x \in A \cap [b, \infty) \\ \text{linear for } x \in A \cap [a,b]. \end{cases}$$

Then $p(f,g) = r/2 < r$, $g \in C(A)$, and $g \notin H(A)$. So $H(A) \cap C(A)$ is nowhere dense in $C(A)$. Since $D(A) \subset H(A)$, the set $D(A) \cap C(A)$ is nowhere dense in $C(A)$.

Remark 1 *There are sets $A \subset \mathbb{R}$ such that the sets $C(A) \cap D(A)$ are not closed in $C(A)$.*

Example 3 *Let $A = [0, 1/2) \cup (1/2, 1]$. Put $f(x) = x$ for $x \in A$. Obviously $f \in C(A) \setminus D(A)$. For $n = 1, 2, \dots$ let $a_n = 2^{-1} - 4^{-n}$ and $b_n = (a_n + 2^{-1})/2$. Define*

$$f_n(x) = \begin{cases} 2^{-1} & \text{for } x \in [b_n, 2^{-1}) \\ x & \text{for } x \in [0, a_n] \cup (2^{-1}, 1] \\ \text{linear in the interval } [a_n, b_n]. \end{cases}$$

Then all $f_n \in C(A) \cap D(A)$ and the sequence (f_n) uniformly converges to f.

Theorem 4 *If $A \subset \mathbb{R}$ is a nonempty closed set, then $C(A) \cap D(A)$ is closed in $C(A)$.*

Proof. If a sequence of functions $f_n \in C(A) \cap D(A)$ converges uniformly to a function f, then $f \in C(A)$. Assume,to the contrary, that $f \notin D(A)$. Then there are points $a, b \in A$ with $a < b$, $f(a) \neq f(b)$ and

$$c \in (\min(f(a), f(b)), \max(f(a), f(b)))$$

such that $c \notin f([a,b] \cap A)$. We may assume that $f(a) < c < f(b)$. Since the set $[a,b] \cap A$ is compact and f is continuous, the set $f([a,b] \cap A)$ is compact. For $r > 0$ there is a function f_n such that $f_n(a) < c < f_n(b)$ and $|f_n(x) - f(x)| < r$ for every $x \in A$. Since $f_n \in D(A)$, there is a point $d \in A \cap (a,b)$ such that $f_n(d) = c$. Consequently,

$$|f(d) - c| = |f(d) - f_n(d)| < r,$$

and $(c - r, c + r) \cap f([a,b] \cap A) \neq \emptyset$. So c is an accumulation point of the compact set $f([a,b] \cap A)$, and $c \in f([a,b] \cap A)$, contrary to $c \neq f([a,b] \cap A)$.

Theorem 5 *Suppose that a nonempty set $A \subset \mathbb{R}$ is such that $cl A - A$ is not closed. Then the set $C(A) \cap D(A)$ is nowhere dense in $C(A)$.*

Proof. If there are points $a, b \in A$ such that $a < b$ and the cardinality of the set $[a, b] \cap A$ is smaller than continuum, then by Theorem 2 the set $C(A) \cap D(A)$ is nowhere dense in $C(A)$. So we may assume that the set $I \cap A$ has the cardinality of the continuum for every closed interval I with ends belonging to A. Since $\mathrm{cl}\, A - A$ is not closed, there is a point $a \in A$ which is an accumulation point of the set $\mathrm{cl}\, A - A$. Fix $f \in C(A)$ and $0 < r < 1$. From the continuity of f at a it follows that there is an open interval $I \ni a$ such that $\mathrm{osc}_{I \cap A} f < r/8$. Since $a \in A$ and a is an accumulation point of $\mathrm{cl}\, A - A$, there are points $b, d \in I \cap A$ and $u \in \mathrm{cl}\, A - A$ such that $b < u < d$. Let us put

$$g(x) = \begin{cases} f(x) & \text{for } x \in A \cap (-\infty, u) \\ f(x) + 3r/4 & \text{for } x \in A \cap (u, \infty). \end{cases}$$

Evidently, $g \in C(A)$ and $p(f, g) = 3r/4$. Let $h \in C(A)$ be such that $p(g, h) < r/8$. Then $p(f, h) \leq p(f, g) + p(g, h) < 3r/4 + r/8 < r$. We shall show that $h \notin D(A)$. We have

$$\begin{aligned} g(b) &= f(b), & g(d) &= f(d) + 3r/4, \\ h(b) &< f(b) + r/8, & h(d) &> f(d) + 3r/4 - r/8 = f(d) + 5r/8 > \\ & & &> f(b) - r/8 + 5r/8 = f(b) + r/2. \end{aligned}$$

Let c be a number such that $f(b) + r/4 < c < f(b) + r/2$. Then $h(b) < c < h(d)$, and for every $x \in [b, u) \cap A$ we have

$$h(x) < g(x) + r/8 = f(x) + r/8 < f(b) + r/8 + r/8 < c.$$

Moreover, for every $x \in (u, d] \cap A$,

$$h(x) > g(x) - r/8 = f(x) + 3r/4 - r/8 > f(b) - r/8 + 5r/8 = f(b) + r/2 > c.$$

So $c \notin h((b, d) \cap A)$, and consequently $h \notin D(A)$. This completes the proof.

Theorem 6 *If a nonempty set A is such that the set $cl A - A$ is closed and there are not points $a, b \in A$ with $a < b$ and such that the cardinality of the set $(a, b) \cap A$ is smaller than continuum, then the set $C(A) \cap D(A)$ has the nonempty interior in $C(A)$.*

Proof. If $\mathrm{cl}\, A - A = \emptyset$ then A is an interval and $C(A) \subset D(A)$. So, we may assume that $\mathrm{cl}\, A - A \neq \emptyset$. Let $((a_n, b_n))_n$ be a sequence with all components of the open set $\mathbb{R}-(\mathrm{cl}\,A-A)$. From the suppositions of our theorem it follows that $A\subset\bigcup_n(a_n,b_n)$, and every set $A\cap(a_n,b_n)$ is connected. If $A\cap(a_n,b_n)=(a_n,c_n]$ (or $=[c_n,b_n)$),then there is a continuous function $f_n : A\cap(a_n,b_n)\to\mathbb{R}$ such that $f_n(c_n)=0$ and the cluster set

$$K^+(f_n,a_n)=\{y\in\mathbb{R}: \text{ there is a sequence of points } x_k\in A\cap(a_n,b_n)$$
$$\text{with } x_k\searrow a_n \text{ and } f_n(x_k)\to y\}=\mathbb{R}$$
$$(K^-(f_n,b_n)=\{y\in\mathbb{R}: \text{ there is a sequence of points } x_k\in A\cap(a_n,b_n)$$
$$\text{with } x_k\nearrow b_n \text{ and } f_n(x_k)\to y\}=\mathbb{R}).$$

If $(a_n,b_n)\subset A$, then there is a continuous function $f_n : (a_n,b_n)\to\mathbb{R}$ such that

$$K^+(f_n,a_n)=K^-(f_n,b_n)=\mathbb{R}.$$

If $(a_n,b_n)\cap A$ is a singleton set $\{c_n\}$, then we put $f_n(c_n)=0$. If $(a_n,b_n)\cap A=[c_n,d_n]\subset(a_n,b_n)$, then there is a continuous function $f_n : [c_n,d_n]\to\mathbb{R}$ such that $f_n(c_n)=f_n(d_n)=0$ and $f_n([c_n,d_n])=[-n,n]$. Let $f(x)=f_n(x)$ for $x\in(a_n,b_n)\cap A$, $n=1,2,\ldots$. Then $f\in C(A)$ and if $u_n=a_n$ or b_n is an accumulation point of the set A from the left (from the right), then $K^-(f,u_n)=\mathbb{R}$ $(K^+(f,u_n)=\mathbb{R})$. Let $g\in C(A)$ be such that $p(f,g)=r<1$. We shall show that $g\in D(A)$. Let $a,b\in A$ be points such that $a<b$ and $g(a)\neq g(b)$, for example $g(a)<g(b)$. Let us fix a number c with $g(a)<c<g(b)$. If there is not a point $u_i=a_i$ or b_i belonging to $[a,b]$, then $[a,b]\subset A$ and $g|[a,b]$ has the Darboux property. Consequently, there is a point $d\in(a,b)\cap A$ such that $g(d)=c$. In the contrary case, if there is a point $u_i=a_i$ or b_i belonging to $[a,b]$, then there are points $u,v\in(a,b)$ such that $f(u)<c-r$, $f(v)>c+r$ and $[\min(u,v),\max(u,v)]\subset A$. Since $p(f,g)=r<1$, we have $g(u)<c$, $g(v)>c$ and $g/[\min(u,v),\max(u,v)]$ is continuous. Consequently, there is a point $d\in(\min(u,v),\max(u,v))\subset(a,b)$ such that $g(d)=c$. So $g\in D(A)$.

Now, for a nonempty set $A\subset\mathbb{R}$ let

$$C_0(A)=\{g:A\to\mathbb{R};\ \text{there is } f\in C(\mathbb{R}) \text{ such that } f/A=g\}.$$

Remark 2 *If $A\subset\mathbb{R}$ is a nonempty set such that there is a point $a\in clA-A$ which is bilateral accumulation point of A, then $C_0(A)$ is a nowhere dense closed set in $C(A)$.*

Proof. If a sequence of functions $g_n\in C_0(A)$ converges uniformly to a function $g : A\to\mathbb{R}$, then there are functions $f_n\in C(\mathbb{R})$ such that $f_n/A=g_n$ and the sequence of functions $f_n/\mathrm{cl}\,A$, $n=1,2,\ldots$, converges uniformly on $\mathrm{cl}\,A$ to a function $h : \mathrm{cl}\, A \to \mathbb{R}$. Evidently, $h \in C_0(\mathrm{cl}\, A)$ and $h/A = g \in C_0(A)$. So $C_0(A)$ is closed in $C(A)$. For a fixed $f \in C(A)$ and for a fixed $r > 0$ $(r < 1)$ we define

$$g(x) = \begin{cases} f(x) & \text{for } \; x \in (-\infty, a) \cap A \\ f(x) + r/2 & \text{for } \; x \in (a, \infty) \cap A. \end{cases}$$

Then $p(f, g) = r/2 < r$ and $g \in C(A) \setminus C_0(A)$. This completes the proof.

Theorem 7 *If a set $A \subset \mathbb{R}$ containing more than one point is not an interval, then the set $C_0(A) \setminus D(A)$ is dense in $C_0(A)$.*

Proof. Let $a \notin A$ be such that $(-\infty, a) \cap A \neq \emptyset$ and $(a, \infty) \cap A \neq \emptyset$. Given a fixed $f \in C_0(A)$ and $1 > r > 0$ there are continuous functions $g, h \in C(\mathbb{R})$ and points $c, d \in A$ such that $h/A = f$, $p(h, g) < r$, $c < a < d$, and $g/[c, d]$ is linear, non constant. Then $g|A \in C_0(A)$ and $g|A \notin D(A)$, since $g(a) \in (\min(g(c), g(d)), \max(g(c), g(d)))$ and $g(a) \notin g(A \cap [c, d])$.

Remark 3 *Example 1 shows that the set $C_0(A) \cap D(A)$ may be not closed in $C_0(A)$. But if a set $A \subset \mathbb{R}$ is nonempty and closed, then $C_0(A) \cap D(A)$ is closed in $C_0(A)$. This follows from Theorem 3.*

Theorem 8 *If there exist points $a, b \in Cl\, A$ such that $a < b$ and the cardinality of the set $(a, b) \cap A$ is smaller than continuum, then the set $C_0(A) \cap D(A)$ is nowhere dense in $C_0(A)$.*

Proof. Given a fixed $f \in C_0(A)$, there is $g \in C(R)$ such that $g/A = f$. Let $1 > r > 0$ be a number. Define

$$h(x) = \begin{cases} g(x) & \text{for } x \in (-\infty, a] \\ g(x) + c & \text{for } x \in [b, \infty) \\ g(x) + c(x - a)/(b - a) & \text{for } x \in [a, b], \end{cases}$$

where $c \in \mathbb{R}$ is such that $|c| < r/2$ and $g(a) \neq g(b) + c$. We may assume that $g(a) < g(b) + c$. Note that $h/A \in C_0(A)$ and $p(h/A, f) \leq r/2 < r$. Put $s = g(b) + c - g(a)$. Then $s > 0$. We shall prove that every function $k \in C_0(A)$ with $p(k, h/A) < \min(r/2, s/8)$ is not in $D(A)$. Indeed, since $k \in C_0(A)$, there is $\ell \in C(\mathbb{R})$ such that $\ell/A = k$. We may assume that $p(\ell, h) < \min(r/2, s/8)$. From the continuity of ℓ and h at a, b it follows that there are points $u, v \in A$ such that $u < v$, and

$$|\ell(x) - \ell(a)| < s/8, \quad |\ell(y) - \ell(b)| < s/8,$$

$$|h(x) - h(a)| < s/8, \quad |h(y) - h(b)| < s/8$$

for all points $x \in [\min(u,a), \max(u,a)] = I$ and all $y \in [\min(v,b), \max(v,b)] = J$. We have

$$k(x) = 1(x) < h(x) + s/8 < h(a) + s/8 + s/8 = g(a) + s/4,$$

$$k(y) = 1(y) > h(y) - s/8 > h(b) - s/8 - s/8 = g(b) + c - s/4$$

for all $x \in I \cap A$ and all $y \in J \cap A$. Since $g(a) + s/4 < g(b) + c - s/4$ and since the set $[a,b] \cap A$ has cardinality smaller than that of the continuum, there is a number $z \in (g(a)+s/4, g(b)+c-s/4)$ such that $k(x) \neq z$ for every $x \in (u,v) \cap A$. Since $p(k,f) \leq p(k, h/A) + p(h/A, f) < r/2 + r/2 = r$, the proof is finished.

Theorem 9 *If $A \subset \mathbb{R}$ is a nonempty set such that clA is a nondegenerate interval and for every open interval I with $A \cap I \neq \emptyset$ the intersection $I \cap A$ contains a nonempty perfect set, then the set $C_0(A) \cap D(A)$ is dense in $C_0(A)$.*

In the proof of this theorem we apply the following lemma:

Lemma 10 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and let $P \subset (a,b)$, $Q \subset [a,b]$ be nonempty perfect sets such that $P \cap Q = \emptyset$. There is a continuous function $g : [a,b] \to R$ such that $(f+g)([a,b]) = f([a,b]) = (f+g)(P)$ and $g(x) = 0$ for $x \in Q \cup \{a,b\}$.*

Proof of Lemma 1. There is ([1], p. 224) a continuous function $h : P \xrightarrow{\text{onto}} f([a,b])$. Let

$$k(x) = \begin{cases} h(x) \text{ for } x \in P \\ f(x) \text{ for } x \in Q \cup \{a,b\} \\ \text{linear in the closure of all components} \\ \text{of the set } (a,b) - P - Q \end{cases}$$

and

$$g = k - f.$$

The function g satisfies all required conditions.

Proof of Theorem 8. Fix $f \in C_0(A)$ and $1 > r > 0$. There is a function $g \in C(R)$ such that $f = g|A$. We shall prove that there is a function $h \in C_0(A) \cap D(A)$ with $p(f,h) < r$. If $f \in D(A)$, then $f = h$. Assume that $f \notin D(A)$. Since the function g is uniformly continuous on the interval $[-1,1]$, there are points

$$\max(-1, \inf Cl\, A)) = a_{11} < a_{12} < \cdots < a_{1,k(1)} = \min(1, \sup Cl\, A)$$

such that

$$a_{1,i+1} - a_{1i} < 1$$

and

$$\operatorname*{osc}_{[a_{1i},a_{1,i+1}]} g < 4^{-1}r$$

for $i = 1, \ldots, k(1) - 1$. For each $i < k(1)$ there is a nonempty perfect set $P_{1i} \subset (a_{1i}, a_{1,i+1}) \cap A$ which is nowhere dense in A. Consequently, by Lemma 1, there are continuous functions $g_{1i} : [a_{1i}, a_{1,i+1}] \to R$ such that

$$(g + g_{1i})([a_{1i}, a_{1,i+1}]) = g([a_{1i}, a_{1,i+1})] = (g + g_{1i})(P_{1i})$$

and $g_{1i}(a_{1i}) = g_{1i}(a_{1,i+1}) = 0$. Let

$$f_1(x) = \begin{cases} g(x) + g_{1i}(x) & \text{for } x \in [a_{1i}, a_{1,i+1}], \quad i < k(1) \\ g(x) & \text{otherwise.} \end{cases}$$

Evidently, $f_1 \in C(\mathbb{R})$ and $p(g, f_1) < 4^{-1}r$. Since the function f_1 is uniformly continuous on the interval $[-2, 2]$, there are points

$$\max(-2, \inf Cl\, A) = a_{21} < \cdots < a_{2k(2)} = \min(2, \sup Cl\, A)$$

such that

$$a_{2,i+1} - a_{2i} < 2^{-1}, \text{ and } \operatorname*{osc}_{[a_{2i},a_{2,i+1}]} f_1 < 4^{-2}r \quad \text{for } i < k(2)$$

For each $i < k(2)$, there is a nonempty perfect set $P_{2i} \subset (A \cap (a_{2i}, a_{2,i+1})) - \bigcup_{i<k(1)} P_{1i}$ which is nowhere dense in A. By Lemma 1, for each $i < k(2)$ there are continuous functions $g_{2i} : [a_{2i}, a_{2,i+1}] \to \mathbb{R}$ such that $(f_1+g_{2i})([a_{2i}, a_{2,i+1}]) = f_1([a_{2i}, a_{2,i+1}]) = (f_1 + g_{2i})(P_{2i})$ and $g_{2i}(x) = 0$ for

$$x \in \{a_{2i}, a_{2,i+1}\} \cup ([a_{2i}, a_{2,i+1}] \cap \bigcup\{P_{1j} : j < k(1)\}).$$

Let

$$f_2(x) = \begin{cases} f_1(x) + g_{2i}(x) & \text{for } x \in [a_{2i}, a_{2,i+1}], \quad i < k(2) \\ f_1(x) & \text{otherwise.} \end{cases}$$

Evidently, $f_2 \in C(\mathbb{R})$ and $p(f_2, f_1) < 4^{-2}r$. Generally, for $n > 2$, there are points

$$\max(-n, \inf \mathrm{cl}\, A) = a_{n1} < \cdots < a_{nk(n)} = \min(n, \sup \mathrm{cl}\, A)$$

with

$$a_{n,i+1} - a_{ni} < 1/n, \quad \operatorname*{osc}_{[a_{ni},a_{n,i+1}]} f_{n-1} < 4^{-n}r \quad \text{for } i < k(n),$$

nonempty perfect sets $P_{ni} \subset (a_{ni}, a_{n,i+1}) \cap A - \bigcup_{j<n} \bigcup_{i<k(j)} P_{ji}$ which are nowhere dense in A ($i < k(n)$), and continuous function $f_n \in C(\mathbb{R})$ such that $p(f_n, f_{n-1}) < 4^{-n}r$, $f_n(P_{ni}) = f_n([a_{ni}, a_{n,i+1}])$ for $i < k(n)$ and $f_n(x) = f_{n-1}(x)$ for $x \in P_{ji}$ with $j < n$, $i < k(j)$. Since $p(f_n, f_{n-1}) < 4^{-n}r$ and $\sum_{n=1}^{\infty} 4^{-n}r < \infty$, the sequence (f_n) converges uniformly to a function $k \in C(\mathbb{R})$. For every $n = 1, 2, \ldots$ $p(g, f_n) \leq p(g, f_1) + \cdots + p(f_{n-1}, f_n) < r/4 + \cdots + r/4^n$, and consequently

$$p(g, k) \leq r \sum_{n=1}^{\infty} 4^{-n} = r/3 < r.$$

Now, we shall show that $k/A = h \in D(A)$. For given points $a, b \in A$ with $a < b$ and $h(a) \neq h(b)$ (for example, $h(a) < h(b)$) let $c \in (h(a), h(b))$. There are points a_1, b_1 such that $a < a_1 < b_1 < b$ and $k(a_1) < c < k(b_1)$. Since the sequence (f_n) converges uniformly to k, there is an index n such that $f_n(a_1) < c < f_n(b_1)$, $[a, b] \subset (-n, n)$ and $1/n < \min(a_1 - a, b - b_1)$. From the continuity of f_n it follows that there is a point $z \in (a_1, b_1)$ such that $f_n(z) = c$. There is an index $i < k(n)$ such that $z \in [a_{ni}, a_{n,i+1}] \subset (a, b)$. Since $f_n(P_{ni}) = f_n([a_{ni}, a_{n,i+1}])$, there is a point $w \in P_{ni}$ with $f_n(w) = f_n(z) = c$. Consequently, $w \in A \cap (a, b)$, and $h(w) = k(w) = f_n(w) = f_n(z) = c$, since $f_k(w) = f_n(w)$ for $k \geq n$. This completes the proof.

Remark 4 *In our discussion with Dr. T. Natkaniec he remarked that if* $cl A$ *is a nondegenerate interval and if there is* $f \in D(A) \cap C_0(A)$ *which is non constant, then* A *contains a nowhere dense (in* $\mathbb{R}$*) subset having the cardinality of the continuum. Since there exist c-dense (in* $\mathbb{R}$*) sets* A *such that for every set* $B \subset \mathbb{R}$ *of the first category the intersection* $B \cap A$ *is countable (for example, Lusin sets), there are sets* $A \subset \mathbb{R}$ *such that* $cl A = \mathbb{R}$ *and* $A \cap I$ *has the cardinality of the continuum for every open interval* I *and* $C_0(A) \cap D(A)$ *is nowhere dense in* $C_0(A)$.

Problem 1 *Suppose that* $cl A$ *is a nondegenerate interval and for every open interval* I *with* $A \cap I \neq \emptyset$ *the intersection* $A \cap I$ *contains a nowhere dense set having the cardinality of the continuum. Is the set* $D(A) \cap C_0(A)$ *dense in* $C_0(A)$*?*

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