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Basic Convergence Principles for the Kurzweil-Henstock Integral

1. Introduction

The Lebesgue integral on $\mathbb{R}$ is a very special case of the Kurzweil-Henstock integral. The latter by its wider scope allows the Lebesgue convergence theorems to be extended to integrands which are not absolutely integrable. Beyond such extensions the Kurzweil-Henstock integration process yields convergence principles that cannot even be formulated in terms of the Lebesgue theory. Such principles are of central concern here. Some we newly introduce. Others stem from the pioneering work of P.Y. Lee [8,9] and his joint work with T.S. Chew [10,11,12] and also from the investigative studies of R. Gordon [1,2]. Where our work overlaps that of others we improve formulations, extend generality, and eliminate irrelevant or redundant hypotheses.

2. Preliminaries

We begin with a review of relevant concepts involving the Kurzweil-Henstock integral and its differentials. For a detailed exposition see [4,5,6]. A *cell* I is a closed, bounded, nondegenerate interval in $\mathbb{R}$. A *tagged cell* (I,t) is a cell I with *one of its endpoints* t designated as the *tag*. While the ultimate objects of integration are differentials for which the integration process yields a sound definition, the immediate objects of integration are summants which generate the summands in the approximating sums. A *summant* S on a cell K is a real-valued function $S(I,t)$ on the set of all tagged cells (I,t) in K. S is a *cell summant* if its values do not depend on the tag, $S(I,t) = S(I)$.

Each function F on K yields a cell summant ΔF defined by $\Delta F(I) = F(s) - F(r)$ for $I = [r,s]$. Such summants play a key role because they are additive on abutting cells. For S a summant on K and f a function on K the product

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fS is the summant whose value at (I,t) is $f(t)S(I,t)$. Each summant S on K has a lower and an upper integral: $-\infty \le \underline{\int}_K S \le \overline{\int}_K S \le \infty$. To define these integrals we need some preliminary definitions. A *division* $\mathcal{K}$ of K is a finite set of nonoverlapping tagged cells whose union is K. A *gauge* δ on K is a function $\delta(t) > 0$ for all t in K. A tagged cell (I,t) is δ *- fine* if the length of I is less than $\delta(t)$. A δ*-division* is a division whose members are δ-fine. For each division $\mathcal{K}$ of K let $\sum_{\mathcal{K}} S$ be the sum of $S(I,t)$ over the members (I,t) of $\mathcal{K}$. For each gauge δ on K let $\sum_{K\delta} S$ be the infimum, $\sum_K^\delta S$ the supremum, of the sums $\sum_{\mathcal{K}} S$ for all δ-divisions $\mathcal{K}$ of K. Define $\underline{\int}_K S = \sup_\delta \sum_{K\delta} S$ and $\overline{\int}_K S = \inf_\delta \sum_K{}^\delta S$ where the supremum and infimum are taken over all gauges δ on K. If the lower and upper integrals are equal then their common value defines the integral $\int_K S$ with its *value in* $[-\infty,\infty]$. S is *integrable* if its integral exists and is finite.

The integration process described above provides a viable definition of differential as an equivalence class of summants. This definition introduced in [4,5] is constructed as follows.

The summants on K form a linear space $\mathcal{S}$ of functions. If S belongs to $\mathcal{S}$ so does $|S|$. So $\mathcal{S}$ is a Riesz space (vector lattice). The summants S with $\int_K |S| = 0$ form a Riesz ideal $\mathcal{T}$ in $\mathcal{S}$. That is, $\mathcal{T}$ is a linear subspace of $\mathcal{S}$ such that if S belongs to $\mathcal{S}$, T belongs to $\mathcal{T}$, and $|S| \le |T|$ then S belongs to $\mathcal{T}$. So $\mathcal{D} = \mathcal{S}/\mathcal{T}$ is a Riesz space with the linear and lattice operations transferred homomorphically from $\mathcal{S}$ to $\mathcal{D}$. A *differential* on K is any element σ of $\mathcal{D}$. Explicitly σ is an equivalence class $[S]$ of summants S on K under the equivalence $S \sim S'$ defined by $\int_K |S - S'| = 0$. For $\rho = [R]$ and $\sigma = [S]$ we have $\rho + c\sigma = [R + cS]$ for every constant c, and $|\sigma| = [|S|]$ which induce the properties $\rho \wedge \sigma = [R \wedge S]$, $\rho \vee \sigma = [R \vee S]$, $\sigma^+ = [S^+]$, $\sigma^- = [S^-]$. The definitions $\underline{\int}_K \sigma = \underline{\int}_K S$ and $\overline{\int}_K \sigma = \overline{\int}_K S$ of lower and upper integral of σ are effective for $\sigma = [S]$. When these two integrals are equal their common value defines $\int_K \sigma$. σ is integrable if $\int_K \sigma$ exists and is finite. If σ is integrable on K then σ is integrable on every cell contained in K. Each function F on K induces an integrable differential $dF = [\Delta F]$ with $\int_I dF = \Delta F(I)$ for every cell I in K. A differential σ is integrable if and only if $\sigma = dF$ for some function F. For σ integrable such a function F is obtained by letting $F(x) = \int_{[a,x]} \sigma$ with $F(a) = 0$. Every function F on K has its *total variation* given by $\int_K |dF| \le \infty$. A differential σ on K is *summable* if its *variational norm* $n(\sigma) = \overline{\int}_K |\sigma|$ is finite.

The norm n is a Riesz norm: $|\rho| \le |\sigma|$ implies $n(\rho) \le n(\sigma)$. Under norm n the summable differentials on K form a Banach lattice [6]. For 1_E the indicator of a subset E of K the product $1_E\,\sigma$ is effectively defined by $1_E\sigma = [1_E S]$ for $\sigma = [S]$. Effectiveness is obvious because 1_E is bounded, taking only the values $0, 1$. E is σ*-null* if $1_E\sigma = 0$. A condition holds σ*-everywhere*, or at σ*-all* t, in K if it holds on the complement of some σ-null set in K. σ is *tag-null* if each point in K is σ-null. If f is a function defined and finite σ-everywhere on K we can effectively define the product $f\sigma = [gS]$ where $\sigma = [S]$ and g is any function on K such that g is everywhere finite and σ-everywhere equal to f. A *damper* is an everywhere positive function on K. σ is *damper-summable* if $u\sigma$ is summable for some damper u. σ is *dampable* if both $u\sigma$ and $u|\sigma|$ are integrable for some damper u. A net of differentials σ_α *converges in variation* to a differential σ if the variational norm $n(\sigma_\alpha - \sigma) \to_\alpha 0$. This does not require that σ_α and σ be summable. σ_α *converges in damped variation* to σ if $u\sigma_\alpha$ converges in variation to $u\sigma$ for some damper u. This convergence, newly introduced here, is much weaker than convergence in variation. As we shall see, it is closely related to convergence in measure. Being a Riesz convergence [7] it gives unique limits. A cell summant S is *superadditive* if

$$S(I) + S(J) \leq S(I \cup J) \text{ for all abutting cells } I, J. \tag{1}$$

S is *subadditive* if the reverse inequality holds in (1). For S superadditive, $-\infty \leq \sum_{K^\delta} S = \int_K \sigma \leq S(K)$ for every gauge δ. For S subadditive, $S(K) \leq \int_K \sigma = \sum_K{}^\delta S \leq \infty$. Since $|\Delta F|$ is subadditive, the total variation of F is $n(dF) = \int_K |dF| = \sum_K{}^\delta |\Delta F|$. The members W of a set $\mathcal{W}$ of summants are *uniformly summable* if each W is summable and $\sum_K{}^\delta |W| \to \overline{\int}_K |W|$ uniformly for all W in $\mathcal{W}$ as $\delta \to 0$. Explicitly, given $\varepsilon > 0$ there exists a gauge δ such that $\sum_K{}^\delta |W| < \overline{\int}_K |W| + \varepsilon$ for all W in $\mathcal{W}$. The members W of $\mathcal{W}$ are *uniformly equivalent to* 0 ($W \approx 0$) if each $W \sim 0$ and the W's are uniformly summable. That is, $\sum_K{}^\delta |W| \to 0$ uniformly for all W in $\mathcal{W}$ as $\delta \to 0$. The W's are *uniformly integrable* if $W - \Delta w \approx 0$ where $dw = [W]$ for each W in $\mathcal{W}$. These *uniform* conditions are properties of the particular summants, not of the differentials.

3. The general convergence problem.

Our focus is on the following problem: Given the hypothesis

(A) $dF_n = [T_n]$ on $K = [a, b]$ with $F_n(a) = 0$ for all n in the set $\mathbb{N}$ of positive integers, and for each tagged cell (I, t) in K, $T_n(I, t) \to T(I, t)$ as $n \to \infty$

find supplementary conditions that yield a function F on K satisfying the two-fold condition

(B) $F_n(t) \to F(t)$ for all t as $n \to \infty$, and $dF = [T]$.

Our first result is a reformulation (C) of (B) which helps to provide solutions to the convergence problem.

Theorem 1. *Given (A), condition (B) is equivalent to*

(C) Given $\varepsilon > 0$ there exist complementary subsets A, B of K, a gauge δ on K, and a summant $W \geq 0$ on K such that

$$\sum_K{}^{\delta} 1_A |\Delta F_n - T_n| < \varepsilon \text{ for all } n, \tag{2}$$

$$\sum_K{}^{\delta} W < \varepsilon, \tag{3}$$

and

for each δ-fine (I, t) with t in B there exists p in $\mathbb{N}$ such that (4)

$$|\Delta F_n(I) - T_n(I,t)| \leq W(I,t) \text{ for all } n \geq p.$$

Proof. Let (A) and (B) hold. We shall show (C) holds with A null, $B = K$, and W superadditive. Under (B), $\Delta F \sim T$. So given $\varepsilon > 0$ there is a gauge δ for which $\sum_K{}^{\delta} |\Delta F - T| < \varepsilon/2$. Let W be the cell summant defined by $W(I) = \sum_I{}^{\delta} |\Delta F - T| + \Delta h(I)$ where h is an increasing linear function on K with $0 < \Delta h(K) < \varepsilon/2$. Since W is superadditive, $\sum_K{}^{\delta} W \leq W(K) < \varepsilon$. This gives (3). For (I, t) δ-fine $|\Delta F(I) - T(I,t)| \leq \sum_I{}^{\delta}|\Delta F - T| < W(I)$ since $\Delta h(I) > 0$. Thus, since $\Delta F_n - T_n \to \Delta F - T$ under (A) and (B), we get (4) with $B = K$. Conversely let (A) and (C) hold. Given $x \neq a$ in $K = [a, b]$ let $J = [a, x]$. Since $F_n(a) = 0$, $F_n(x) = \Delta F_n(J)$. Given $\varepsilon > 0$ apply (C) to get A, B, δ, W. Take a δ-division $\mathcal{J}$ of J. Then $|F_m(x) - F_n(x)| = |\Delta F_m - \Delta F_n|(J) \leq \sum_{\mathcal{J}} |\Delta F_m - T_m| + \sum_{\mathcal{J}} |T_m - T_n| + \sum_{\mathcal{J}} |T_n - \Delta F_n|$. This gives $\overline{\lim}_{m,n\to\infty} |F_m(x) - F_n(x)| < 4\varepsilon$ since $\lim_{m,n\to\infty} \sum_{\mathcal{J}} |T_m - T_n| = 0$ because $T_n \to T$, and

$$\overline{\lim}_{n\to\infty} \sum_{\mathcal{J}} |\Delta F_n - T_n| \leq \varepsilon + \sum_{\mathcal{J}} W < 2\varepsilon \tag{5}$$

by (2), (4), (3). So the Cauchy criterion for the convergence of $F_n(x)$ holds. This lets us define $F(x) = \lim_{n\to\infty} F_n(x)$ for all x in K. Thus $\Delta F_n - T_n$ converges to $\Delta F - T$. So (5) for the case $x = b$, $J = K$ gives $\sum_{\mathcal{K}} |\Delta F - T| < 2\varepsilon$ for every δ-division $\mathcal{K}$ of K. Thus $\Delta F \sim T$. That is, $dF = [T]$. □

For an equivalent formulation of (C) in Theorem 1 we can replace (3) by the usual demand that W be superadditive and $W(K) < \varepsilon$. Indeed, if we define $W^*(I) = \sum_I{}^{\delta} W$ we get a superadditive $W^* \geq 0$ with $W \leq W^*$ at δ-fine tagged cells and $W^*(K) = \sum_K{}^{\delta} W < \varepsilon$. The proof of Theorem 1 shows that under (A) the special case A null, $B = K$ of (C) is actually equivalent to (C). At the other extreme the special case $A = K$, B null gives the following elementary but important solution of the convergence problems.

Theorem 2. *If (A) holds and the T_n's are uniformly integrable then (B) holds.*

Proof. Apply Theorem 1 with $A = K$, B null, $W = 0$, and the conclusion that (A), (C) with (4) vacuous imply (B). □

The role of uniform integrability in convergence theorems has been studied by R. Gordon [2] for the case $T_n(I,t) = f_n(t)\Delta x(I)$ with $x(t) = t$, the identity on K. Uniform integrability lies at the base of the controlled convergence theorem [10].

4. The convergence problem under hypothesis (D).

Throughout this section we assume the following hypothesis.

(D) Let σ be a damper-summable differential on $K = [a,b]$. For all n in $\mathbb{N}$ let $dF_n = f_n\sigma$ with $F_n(a) = 0$ and $f_n \to f$ σ-everywhere on K.

Let $\mathbb{N}^K$ be the set of all positive integer-valued functions N on K. Using this partially ordered set to index nets of summants and their differentials we can formulate some solutions of the convergence problem. The main source of our results is the following definition.

$$\text{For each } N \text{ in } \mathbb{N}^K \text{ let } \rho_N = [R_N] \text{ with } R_N(I,t) = \Delta F_n(I) \text{ for } n = N(t). \quad (6)$$

For convenience we shall combine the proofs of our next two theorems.

Theorem 3. *Let (D) hold. Then ρ_N converges in variation to $f\sigma$ as $N \to \infty$ in $\mathbb{N}^K$.*

Theorem 4. *Let (D) hold. For each $\varepsilon > 0$ let there exist a gauge δ on K, P in $\mathbb{N}^K$, and a summant $U \geq 0$ on K such that $\sum_K{}^\delta U < \varepsilon$ and*

$$|\Delta(F_m - F_n)(I)| \leq U(I,t) \text{ for } (I,t) \ \delta\text{-fine and } m,n \geq P(t). \quad (7)$$

Then there is a function F on K such that $F_n \to F$ as $n \to \infty$ in $\mathbb{N}$, and $dF = f\sigma$.

Proof. Let $\varepsilon > 0$ be given. To prove Theorem 3 we shall find P in $\mathbb{N}^K$ such that

$$n(\rho_N - f\sigma) < \varepsilon \text{ for all } N \geq P \text{ in } \mathbb{N}^K. \quad (8)$$

Since σ is damper-summable there is a damper w small enough so that

$$n(w\sigma) < \varepsilon/2. \quad (9)$$

By annihilating f_n and f on a σ-null set we may assume $f_n \to f$ everywhere on K. This has no effect on the differentials $f_n\sigma$ or $f\sigma$. Choose P in $\mathbb{N}^K$ large enough so that for each t in K

$$|f_n(t) - f(t)| < w(t) \text{ for all } n \geq P(t). \tag{10}$$

We contend that P satisfies (8). Choose a summant S representing σ. Then $\Delta F_n \sim f_n S$ since $dF_n = f_n\sigma$ by (D). So for each n in $\mathbb{N}$ we can choose a gauge δ_n such that $\sum_K^{\delta_n} |\Delta F_n - f_n S| < \varepsilon/2^{n+1}$. Define $V_n(I) = \sum_I^{\delta_n} |\Delta F_n - f_n S|$ and $V(I) = \sum_{n\in\mathbb{N}} V_n(I)$ for each cell I in K. Since V_n is superadditive and $0 \leq V_n < \varepsilon/2^{n+1}$ we can conclude that

$$V \text{ is superadditive and } V(K) < \varepsilon/2. \tag{11}$$

Moreover,

$$|\Delta F_n - f_n S| \leq V \text{ at } \delta_n\text{-fine tagged cells.} \tag{12}$$

For each N in $\mathbb{N}^K$ define the functions $\delta_N > 0$ and f_N on K by

$$\delta_N(t) = \delta_n(t) \text{ and } f_N(t) = f_n(t) \text{ with } n = N(t). \tag{13}$$

Using (6) and (13) we reformulate (12) as (14).

$$|R_N - f_N S| \leq V \text{ at } \delta_N\text{-fine tagged cells.} \tag{14}$$

For $N \geq P$ in $\mathbb{N}^K$ (14) and (10) give $|R_N - fS| \leq |R_N - f_N S| + |f_N - f||S| \leq V + w|S|$ at δ_N-fine tagged cells. Taking upper integrals we get $n(\rho_N - f\sigma) \leq V(K) + n(w\sigma) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (11) and (9). So (8) holds giving Theorem 3.

To prove Theorem 4 we contend that there is a summant $W \geq 0$, P in $\mathbb{N}^K$, and a gauge δ on K such that $\sum_K {}^\delta W < 2\varepsilon$ and

$$|\Delta F_n(I) - f_n(t)S(I,t)| \leq W(I,t) \text{ at } \delta\text{-fine } (I,t) \text{ for all } n \geq P(t). \tag{15}$$

Then Theorem 4 will follow from Theorem 1 applied with A null, $B = K$, and $T_n = f_n S$. We may assume that P in Theorem 4 is large enough to satisfy (10), and δ small enough so that $\delta < \delta_P$ with δ_P given by (13), and by (9)

$$\sum_K {}^\delta w|S| < \varepsilon/2. \tag{16}$$

Then for (I,t) δ-fine, $p = P(t)$, and $n \geq p$

$$\begin{aligned} |\Delta F_n - f_n S| &\leq |\Delta F_n - \Delta F_p| + |\Delta F_p - f_p S| + |f_p - f_n||S| \\ &\leq U + V + 2w|S| \text{ at } (I,t) \end{aligned} \tag{17}$$

by (7) with $m = p$, (12) with $n = p$, and (10). Let

$$W = U + V + 2w|S|. \tag{18}$$

Then $\sum_K {}^\delta W \leq \sum_K {}^\delta U + V(K) + 2\sum_K {}^\delta w|S| < \varepsilon + \varepsilon/2 + \varepsilon/2 = 2\varepsilon$ by the hypothesis on U, (11), and (16). (15) follows from (17) and (18). □

Definition (6) leads to the following concise reformulation of Theorem 4.

Theorem 5. *If (D) holds and there exists P in $\mathbb{N}^K$ such that $R_M - R_N$ is uniformly summable for all $M, N \geq P$ in $\mathbb{N}^K$, then there is a function F on K such that $F_n \to F$ and $dF = f\sigma$.*

Proof. Let $\varepsilon > 0$ be given. By Theorem 3 there exists P in $\mathbb{N}^K$ with

$$n(\rho_M - \rho_N) < \varepsilon/8 \text{ for all } M, N \geq P \text{ in } \mathbb{N}^K. \tag{19}$$

By the hypothesis of uniform summability there is a gauge δ such that for some P in $\mathbb{N}^K$ large enough to satisfy (19),

$$\sum_K {}^\delta |R_M - R_N| < n(\rho_M - \rho_N) + \varepsilon/8 \text{ for all } M, N \geq P. \tag{20}$$

By (19) and (20),

$$\sum_K {}^\delta |R_M - R_N| < \varepsilon/4 \text{ for all } M, N \geq P. \tag{21}$$

Let $Q_+(I,t) = 1$ if t is the left endpoint of $I, 0$ if t is the right endpoint. Let $Q_- = 1 - Q_+$. Q_+ indicates that I is situated at $t+$. Q_- indicates that I is situated at $t-$. Define the cell summant U_+ by

$$U_+(I) = \sup_{M,N\geq P} \sum_I {}^\delta Q_+ |R_M - R_N| \tag{22}$$

with a similar definition for U_-, replacing $+$ by $-$ in (22). Let $U = U_+ + U_-$. By (21) and (22), $U_+(K) \leq \varepsilon/4$. Similarly $U_-(K) \leq \varepsilon/4$. So $U(K) \leq \varepsilon/2 < \varepsilon$. We contend that U_+, U_- and hence U are superadditive. Let cells $I = [r,s]$ and $J = [s,t]$ abut at s with union $L = [r,t]$. Consider any δ-divisions $\mathcal{I}$ of I and $\mathcal{J}$ of J. Then $\mathcal{I} \cup \mathcal{J}$ is a δ-division $\mathcal{L}$ of L. Given $M_1, M_2, N_1, N_2 \geq P$ let $M = M_1$ and $N = N_1$ on $L \setminus J$, $M = M_2$ and $N = N_2$ on J. Then

$$\sum_{\mathcal{I}} Q_+ |R_{M_1} - R_{N_1}| + \sum_{\mathcal{J}} Q_+ |R_{M_2} - R_{N_2}| = \sum_{\mathcal{L}} Q_+ |R_M - R_N|. \tag{23}$$

By (22) and (23), $U_+(I)+U_+(J) \le U_+(L)$. So U_+ is superadditive. By a similar proof so is U_-. Hence U is superadditive. Thus $\sum_K {}^\delta U \le U(K) < \varepsilon$. Moreover, $|R_M - R_N| = Q_+|R_M - R_N| + Q_-|R_M - R_N| \le U_+ + U_- = U$ at δ-fine tagged cells for $M, N \ge P$. This gives (7) by (6). So Theorem 4 gives Theorem 5. □

We remark that in the converse direction the hypothesis of Theorem 4 implies the uniform summability condition in the hypothesis of Theorem 5. Indeed, (7) gives $|R_M - R_N| \le U$ at all δ-fine tagged cells for $M, N \ge P$. So $\sum_K {}^\delta |R_M - R_N| \le \sum_K {}^\delta U < \varepsilon$. Thus $\sum_K {}^\delta |R_M - R_N| < n(\rho_M - \rho_N) + \varepsilon$ for $M, N \ge P$, giving the uniform summability of $R_M - R_N$ for $M, N \ge P$.

In Theorem 4 we may assume that U is superadditive and $U(K) < \varepsilon$. (See our remark at the end of the proof of Theorem 1.)

A special case of Theorem 4 is Corollary 8.15 on p.48 in [9] where Lee treats the case $\sigma = dx$ and $U = \varepsilon \Delta x$ for x the identity $x(t) = t$. This makes the F_n's equidifferentiable dx-everywhere since at dx-all t we have $F_n'(t) = f_n(t)$ and $f_n(t) \to f(t)$ as $n \to \infty$.

Our next two theorems are concerned exclusively with the integrablity of $f\sigma$ under (D). The question of convergence of the functions $F_1, \ldots, F_n, \ldots$ is not addressed. Theorem 6 treats the integrability of $f\sigma$ in terms of its primitives, $dF = f\sigma$. Theorem 7 does not involve F.

Theorem 6. *Given (D) and a function F on K the following six conditions are equivalent:*

(i) $dF = f\sigma$,

(ii) *ρ_N converges in variation to dF as $N \to \infty$ in $\mathbb{N}^K$,*

(iii) $\underline{\lim}\, n(\rho_N - dF) = 0$ *as* $N \to \infty$ *in* $\mathbb{N}^K$,

(iv) *ρ_N converges in damped variation to dF as $N \to \infty$ in $\mathbb{N}^K$,*

(v) *given $\varepsilon > 0$ and P in $\mathbb{N}^K$ there exist $N \ge P$ in $\mathbb{N}^K$ and a gauge δ on K such that*

$$\sum_K {}^\delta |R_N - \Delta F| < \varepsilon, \tag{24}$$

(vi) *given a cell J in K, $\varepsilon > 0$, and P in $\mathbb{N}^K$ there exist $N \ge P$ in $\mathbb{N}^K$ and a gauge δ on K such that*

$$|\Delta F(J) - \sum_{\mathcal{J}} R_N| < \varepsilon \text{ for every } \delta\text{-division } \mathcal{J} \text{ of } J. \tag{25}$$

Proof. The equivalence of each of the conditions (ii), (iii), and (iv) with (i) follows directly from Theorem 3. (v) is just an explicit formulation of (iii). So we need only prove that (v) implies (vi) and that (vi) implies (i). To show that (v) implies (vi) let J, ε, P be given. Apply (v) to get N and δ. For $\mathcal{J}$ any δ-division of J (24) gives $|\Delta F(J) - \sum_{\mathcal{J}} R_N| = |\sum_{\mathcal{J}} \Delta F - R_N| \leq \sum_J{}^\delta |\Delta F - R_N| \leq \sum_K{}^\delta |\Delta F - R_N| < \varepsilon$. So (25) holds, giving (vi). To prove (vi) implies (i) let J, ε be given. By Theorem 3 we may assume that P provided by (vi) is large enough so that

$$n(f\sigma - \rho_N) < \varepsilon \text{ for all } N \geq P \text{ in } \mathbb{N}^K. \tag{26}$$

Condition (vi) gives some $N \geq P$ such that by (25)

$$\Delta F(J) - \varepsilon \leq \underline{\int}_J \rho_N \leq \overline{\int}_J \rho_N \leq \Delta F(J) + \varepsilon. \tag{27}$$

By (26) and (27), $\Delta F(J) - 2\varepsilon \leq \underline{\int}_J \rho_N - \varepsilon < \underline{\int}_J \rho_N + \overline{\int}_J f\sigma - \rho_N \leq \underline{\int}_J f\sigma \leq \overline{\int}_J f\sigma \leq \overline{\int}_J f\sigma - \rho_N + \overline{\int}_J \rho_N < \varepsilon + \overline{\int}_J \rho_N \leq \Delta F(J) + 2\varepsilon$. That is, $\Delta F(J) - 2\varepsilon < \underline{\int}_J f\sigma \leq \overline{\int}_J f\sigma < \Delta F(J) + 2\varepsilon$. Since ε is arbitrary, $\int_J f\sigma = \Delta F(J)$ for every cell J in K. That is, $f\sigma = dF$ by Theorem 5 in [5]. □

On $\mathbb{R}$ Theorem 6 generalizes a result of P.Y. Lee (Lemma 1 on p.99 in [8], Theorems 9.3 and 21.4 and Corollary 9.4 in [9].) Lee considers only the case $\sigma = dx$ for x the identity function on K. Moreover, his hypothesis demands the convergence $F_n \to F$ which according to our Theorem 6 is irrelevant. Lee's conditions seem to be awkward and ambiguous formulations of our conditions (v) and (vi) in Theorem 6.

Theorem 7. *Let (D) hold. Then $f\sigma$ is integrable if and only if the lower and upper integrals of ρ_N over K converge to the same finite limit as $N \to \infty$ in $\mathbb{N}^K$.*

Proof. Given $\varepsilon > 0$ apply Theorem 3 to get P in $\mathbb{N}^K$ satisfying (26). Then for all $N \geq P$ in $\mathbb{N}^K$ we have

$$\begin{aligned} -\varepsilon < -n(f\sigma - \rho_N) &\leq \underline{\int}_K (f\sigma - \rho_N) \leq \overline{\int}_K (f\sigma - \rho_N) \\ &\leq n(f\sigma - \rho_N) < \varepsilon. \end{aligned} \tag{28}$$

So

$$\begin{aligned} \underline{\int}_K \rho_N - \varepsilon &\leq \underline{\int}_K \rho_N + \underline{\int}_K (f\sigma - \rho_N) \leq \underline{\int}_K f\sigma \\ &\leq \overline{\int}_K f\sigma \leq \overline{\int}_K (f\sigma - \rho_N) + \overline{\int}_K \rho_N \leq \varepsilon + \overline{\int}_K \rho_N. \end{aligned} \tag{29}$$

If the lower and upper integrals of ρ_N over K converge to the same finite limit c as $N \to \infty$ in $\mathbb{N}^K$ then (29) gives $c - \varepsilon \leq \underline{\int}_K f\sigma \leq \overline{\int}_K f\sigma \leq \varepsilon + c$ for all $\varepsilon > 0$. So $\int_K f\sigma = c$. Conversely this implies $\underline{\int}_K \rho_N = \underline{\int}_K (\rho_N - f\sigma) + c$ and $\overline{\int}_K \rho_N = \overline{\int}_K (\rho_N - f\sigma) + c$. So by (28) $c - \varepsilon \leq \underline{\int}_K \rho_N \leq \overline{\int}_K \rho_N \leq c + \varepsilon$ for all $N \geq P$. That is, both $\underline{\int}_K \rho_N$ and $\overline{\int}_K \rho_N$ converge to the finite limit c as $N \to \infty$ in $\mathbb{N}^K$. □

5. Hypothesis (D) with $\sigma = dg$ for g continuous, $|dg|$ dampable

Another solution of the convergence problem is suggested by the following formulation of the fundamental theorem of calculus (Theorem 17 in [5]): Let g, f, F be functions on K with g continuous and $|dg|$ dampable. Then $dF = fdg$ if and only if $\frac{dF}{dg}(t) = f(t)$ at dg-all t, and every dg-null set is dF-null.

Theorem 8. *Let g be a continuous function on $K = [a, b]$ with $|dg|$ dampable. Let $dF_n = f_n dg$ on K with $F_n(a) = 0$ for all n in $\mathbb{N}$. Let A, B be complementary subsets of K such that A is dg-null, $1_A \Delta F_n$ is uniformly equivalent to 0 for all n, $f_n \to f$ on B, and given a damper w on K there is a gauge δ on K such that for each δ-fine (I, t) with t in B there exists p in $\mathbb{N}$ satisfying*

$$|\Delta F_n(I) - f_n(t)\Delta g(I)| \leq w(t)|\Delta g(I)| \text{ for all } n \quad (30)$$

$$\text{such that } n \geq p.$$

Then there is a function F on K such that $F_n(t) \to F(t)$ as $n \to \infty$ for all t in K, and $dF = fdg$.

Proof. Given $\varepsilon > 0$ choose a damper w such that

$$\int_K w|dg| < \varepsilon. \quad (31)$$

Choose δ as hypothesized. Since $1_A|\Delta F_n|$ integrates uniformly to 0 we may assume that δ is also small enough so that

$$\sum_K^{\delta} 1_A |\Delta F_n| < \varepsilon \text{ for all } n \quad (32)$$

and by (31)

$$\sum_K^{\delta} w|\Delta g| < \varepsilon. \quad (33)$$

Apply Theorem 1 with $T_n(I,t) = 1_B(t)f_n(t)\Delta g(I)$ and $W(T,t) = w(t)|\Delta g(I)|$. Since $1_A T_n = 0$ (32) is just (2) in Theorem 1. (33) is (3), and (30) gives (4). $T_n \to T$ with $T = 1_B f \Delta g$. So Theorem 1 gives Theorem 8. □

Note that (30) can be replaced by the inequality

$$\overline{\lim}_{n\to\infty}|\Delta F_n(I) - f(t)\Delta g(I)| \leq w(t)|\Delta g(I)| \quad \text{for all } \delta\text{-fine}$$

(I,t) with t in B. This condition is necessary for the conclusion of Theorem 8. The uniform integrability to 0 of $1_A|\Delta F_n|$ in Theorem 8 is a reasonable condition to demand. Our next theorem shows this.

Theorem 9. (a) *If $S_n(I,t) \to S(I,t)$ as $n \to \infty$ for each tagged cell (I,t) in K, and $1_A S_n$ is uniformly equivalent to* 0 *for all n in $\mathbb{N}$, then $1_A S$ is equivalent to* 0.

(b) *If dF_n converges in variation to dF and $1_A dF_n = 0$ for all n, then $1_A \Delta F_n$ is uniformly equivalent to* 0 *for all n.*

Proof. To prove (a) just apply Theorem 2 with $T_n = 1_A S_n$ and $T = 1_A S$. Given the hypothesis in (b) we contend that $1_A|\Delta F_n|$ integrates uniformly to 0, that is, given $\varepsilon > 0$ there is a gauge δ such that for all n in $\mathbb{N}$

$$\sum_K{}^{\delta} 1_A|\Delta F_n| < \varepsilon. \tag{34}$$

To find such a δ use the given convergence $n[d(F_n - F)] \to 0$ to choose p in $\mathbb{N}$ large enough so that

$$n[d(F_n - F)] < \varepsilon/2 \text{ for all } n > p. \tag{35}$$

Then take δ small enough so that both

$$\sum_K{}^{\delta} 1_A|\Delta F_n| < \varepsilon/2 \text{ for } n = 1,\ldots,p \tag{36}$$

and

$$\sum_K{}^{\delta} 1_A|\Delta F| < \varepsilon/2. \tag{37}$$

Such δ exist for (36) because $1_A dF_n = 0$, and for (37) because $1_A dF = 0$ which is implied by $|1_A dF| = 1_A|dF - dF_n| \leq |dF - dF_n| \to 0$ in variation. Now $\sum_K{}^{\delta} 1_A|\Delta F_n| \leq \sum_K{}^{\delta} 1_A|\Delta F_n - \Delta F| + \sum_K{}^{\delta} 1_A|\Delta F| \leq \sum_K{}^{\delta} |\Delta F_n - \Delta F| + \varepsilon/2 = n[d(F_n - F)] + \varepsilon/2$ by (37). This gives (34) for $n > p$ by (35). (34) holds for $n \leq p$ by (36). So (34) holds for all n in $\mathbb{N}$. □

6. Sigma-convergence in variation of dF_n with sigma-uniform summability of $\Delta(F_m - F_n)$

Our next result deals exclusively with the additive summants ΔF_n and the differentials dF_n they represent for a given sequence of functions F_n on K.

Theorem 10. *Let $F_1, \ldots, F_n, \ldots$ be a sequence of functions on $K = [a,b]$ with $F_n(a) = 0$. Let $E_1, \ldots, E_i, \ldots$ be a sequence of disjoint subsets of K covering K such that*

$$\text{given } i \text{ in } \mathbb{N} \text{ and } \varepsilon > 0 \text{ there exist } p \text{ in } \mathbb{N} \text{ and a gauge } \delta \text{ on } K \text{ such that for all } m, n \geq p \quad \sum_K {}^{\delta} 1_{E_i} |\Delta(F_m - F_n)| < \varepsilon. \tag{38}$$

Then there is a function F on K such that $F_n(t) \to F(t)$ as $n \to \infty$ for all t in K. Moreover, for each i in $\mathbb{N}$

$$1_{E_i} dF_n \text{ converges in variation to } 1_{E_i} dF \text{ as } n \to \infty. \tag{39}$$

Finally if there is also some p in $\mathbb{N}$ such that

$$1_{E_i} d(F_m - F_n) \text{ is summable for all } m, n \geq p \text{ and all } i \tag{40}$$

then

$$dF_n \text{ converges in damped variation to } dF. \tag{41}$$

Proof. Given $\varepsilon > 0$ let $\varepsilon_i = \varepsilon/2^i$. So $\sum_{i \in N} \varepsilon_i = \varepsilon$. Apply (38) with $\varepsilon = \varepsilon_i$ to get $p = p_i$ and $\delta = \delta_i$. Since in (38) δ_i is operative only on E_i we can replace δ_i by δ defined by $\delta(t) = \delta_i(t)$ for t in E_i. Given a cell J in K take a δ-division $\mathcal{J}$ of J. Take k in $\mathbb{N}$ large enough so that all of the tags t for (I,t) in $\mathcal{J}$ belong to $E_1 \cup \cdots \cup E_k$. Let q be the largest of the integers $p_1, \ldots, p_k$. Then for $m, n \geq q$ (38) gives $|\Delta(F_m - F_n)J| \leq \sum_{\mathcal{J}} |\Delta(F_m - F_n)| = \sum_{i=1}^k \sum_{\mathcal{J}} 1_{E_i} |\Delta(F_m - F_n)| \leq \sum_{i=1}^k \sum_K {}^{\delta} 1_{E_i} |\Delta(F_m - F_n)| < \sum_{i=1}^k \varepsilon_i < \varepsilon$. That is, given a cell J and $\varepsilon > 0$ there exists q in $\mathbb{N}$ such that $|\Delta F_m(J) - \Delta F_n(J)| < \varepsilon$ for all $m, n \geq q$. This is just the Cauchy criterion for the convergence of $\Delta F_n(J)$ as $n \to \infty$. For $J = [a,t]$ we have $\Delta F_n(J)$ equal to $F_n(t)$ since $F_n(a) = 0$. So we can define $F(t) = \lim_{n \to \infty} F_n(t)$. For any δ-division $\mathcal{K}$ of K (38) gives for all $m, n \geq p_i$, $\sum_{\mathcal{K}} 1_{E_i} |\Delta(F_m - F_n)| < \varepsilon_i$. As $m \to \infty$ this gives $\sum_{\mathcal{K}} 1_{E_i} |\Delta(F - F_n)| \leq \varepsilon_i$ for all $n \geq p_i$ since $F_m \to F$. So $\sum_{\mathcal{K}} 1_{E_i} |\Delta(F - F_n)| \leq \varepsilon_i < \varepsilon$ for all δ-divisions $\mathcal{K}$ of K and all $n \geq p_i$. Taking upper integrals we get

$$n(1_{E_i} |dF - dF_n|) \leq \varepsilon_i < \varepsilon \text{ for all } n \geq p_i. \tag{42}$$

This gives (39). If we also have (40) then $1_{E_i}d(F-F_n)$ is summable for all $n \geq p$ and all i since $n[1_{E_i}d(F-F_n)] \leq n[1_{E_i}d(F-F_m)] + n[1_{E_i}d(F_m - F_n] < \infty$ for m exceeding p_i and p, by (42) and (40). Now (39) is just

$$n[1_{E_i}d(F-F_n)] \to 0 \text{ as } n \to \infty. \tag{43}$$

So for each i we can choose c_i in $(0, \infty)$ such that

$$n[1_{E_i}d(F-F_n)] < c_i \text{ for all } n \geq p. \tag{44}$$

Take a_i in $(0,1)$ such that $\sum_{i\in\mathbb{N}} a_i c_i < \infty$. Define the damper u by $u(t) = a_i$ for t in E_i. Given $\alpha > 0$ take k in $\mathbb{N}$ large enough so that $\sum_{i=k}^{\infty} a_i c_i < \alpha$. Then since $u = \sum_{i\in\mathbb{N}} a_i 1_{E_i}$ Theorem 7 of [5] gives

$$\begin{aligned} n[ud(F-F_n)] &\leq \sum_{i\in\mathbb{N}} n[a_i 1_{E_i} d(F-F_n)] \\ &= \sum_{i\in\mathbb{N}} a_i n[1_{E_i}d(F-F_n)] \\ &\leq \sum_{i=1}^{k} n[1_{E_i}d(F-F_n)] + \sum_{i=k}^{\infty} a_i c_i \\ &< \sum_{i=1}^{k} n[1_{E_i}d(F-F_n)] + \alpha \end{aligned}$$

for $n \geq p$ by (44). Hence by (43) $\overline{\lim}_{n\to\infty} n[ud(F-F_n)] \leq \alpha$. This gives (41) since α is arbitrary. □

The condition (38) in Theorem 10 is called "generalized $\mathcal{P}$-Cauchy" by R. Gordon [2]. Gordon treats only the case $dF_n = f_n dx$ for x the identity function and $f_n \to f$. His hypothesis demands that $F_n \to F$. Our Theorem 10 reveals that this assumption is redundant. (38) is equivalent to the twofold condition: For each i in $\mathbb{N}$, $1_{E_i}dF_n$ converges in variation as $n \to \infty$, and there exists p_i in $\mathbb{N}$ such that the summants $1_{E_i}\Delta(F_m - F_n)$ with $m, n \geq p_i$ are uniformly summable. P.Y. Lee introduced a condition he calls "oscillation convergence" which implies (38). (See Definition 9.5 in [9].)

7. Convergence in damped variation and convergence in measure.

The concept of convergence in measure with respect to a summable differential σ is motivated by the following theorem.

Theorem 11. *Let σ be a summable differential on K. Let $g_1, \ldots, g_n, \ldots$ be a sequence of nonnegative functions on K. Then the following three conditions are equivalent:*

(a) *If $\varepsilon > 0$ and E_n is the set of all t at which $g_n(t) > \varepsilon$ then $n(1_{E_n}\sigma) \to 0$ as $n \to \infty$,*

(b) *$n(g_n\sigma/1 + g_n) \to 0$ as $n \to \infty$,*

(c) *$n(1 \wedge g_n\ \sigma) \to 0$ as $n \to \infty$.*

Proof. Given $\varepsilon > 0$ let $E_n = g_n^{-1}(\varepsilon, \infty)$. The inequalities $\frac{\varepsilon}{1+\varepsilon} 1_{E_n} \leq \frac{g_n}{1+g_n} \leq 1 \wedge g_n \leq \varepsilon + 1_{E_n}$ are easily verified. They imply $\frac{\varepsilon}{1+\varepsilon} n(1_{E_n}\sigma) \leq n\left(\frac{g_n}{1+g_n}\sigma\right) \leq n(1 \wedge g_n\ \sigma) \leq \varepsilon n(\sigma) + n(1_{E_n}\sigma)$. Taking upper limits as $n \to \infty$ we get the equivalence of (a), (b), and (c). □

For a sequence of functions f_n on K and σ a summable differential on K we say that f_n *converges in measure* to f with respect to σ if any, hence all, of the conditions (a), (b), (c) hold for $g_n = |f - f_n|$. This concept is closely related to convergence in damped variation of $f_n\sigma$ to $f\sigma$. Let us investigate this relationship in our next two theorems.

Theorem 12. *Let σ be a summable differential on K. Let $v, f, f_1, \ldots, f_n, \ldots$ be functions on K such that $|f_n| \leq v$ for all n σ-everywhere, and f_n converges in measure to f with respect to σ. Then $f_n\sigma$ converges in damped variation to $f\sigma$.*

Proof. Let $w = 1/1+v+|f|$ and $g_n = |f_n - f|$. Since $g_n \leq |f_n| + |f| \leq v + |f|$ σ-everywhere, $1 + g_n \leq 1 + v + |f|$ σ-everywhere. So $w \leq 1/1 + g_n$ σ-everywhere. Clearly $w(t) > 0$ for all t in K. Also $wg_n|\sigma| \leq g_n|\sigma|/1 + g_n$. By the convergence in measure criterion (b) in Theorem 11 this gives the convergence in variation to 0 of $wg_n|\sigma|$, that is, the convergence in variation of $wf_n\sigma$ to $wf\sigma$. So $f_n\sigma$ converges in damped variation to $f\sigma$. □

(For the measure induced by a summable σ see [6] Prop. 8.)

We can get a converse to Theorem 12 if σ is absolutely integrable and the damper is measurable with respect to σ. For $dg \geq 0$ on K a function w on K is *dg-measurable* if $1_E dg$ is integrable for each set E of the form $w^{-1}(c, \infty)$ with c in $\mathbb{R}$. dg-measurability is preserved under the usual algebraic and sequential limit operations on dg-measurable functions [5].

Theorem 13. *Let $dg \geq 0$ on K, and $wf_n dg$ converge in variation to $wfdg$ for some dg-measurable damper w on K. Then f_n converges in measure to f with respect to dg.*

Proof. Let $g_n = |f_n - f|$. Given $\varepsilon > 0$ let $E_n = g_n^{-1}(\varepsilon, \infty)$. We contend

$$n(1_{E_n} dg) \to 0 \text{ as } n \to \infty \tag{45}$$

thereby giving (a) of Theorem 11. Clearly

$$1_{E_n} \leq (1 - mw)^+ + mw 1_{E_n} \text{ for all } m, n \text{ in } \mathbb{N}. \tag{46}$$

Since w is dg-measurable, $(1 - mw)^+ dg$ is integrable. So by (46)

$$n(1_{E_n} dg) \leq \int_K (1 - mw)^+ dg + mn(w 1_{E_n} dg). \tag{47}$$

By the definition of E_n, $\varepsilon 1_{E_n} \leq g_n$. So $\varepsilon n(w 1_{E_n} dg) \leq n(wg_n dg)$. The last term goes to 0 as $n \to \infty$ by hypothesis. So $n(w 1_{E_n} dg) \to 0$ as $n \to \infty$. Hence (47) yields

$$\underline{\lim}_{n \to \infty} n(1_{E_n} dg) \leq \int_K (1 - mv)^+ dg \text{ for all } m \text{ in } \mathbb{N}. \tag{48}$$

Since $w(t) > 0$ for all t in K, $(1 - mv)^+ \searrow 0$ as $m \nearrow \infty$. Thus

$$\int_K (1 - mw)^+ dg \searrow 0 \text{ as } m \nearrow \infty \tag{49}$$

by the bounded convergence theorem. (48), (49) give (45). □

Theorem 14. *Let $dg \geq 0$ on K and $f_1, \ldots, f_n, \ldots$ be dg-measurable functions on K with $f_n \to f$ dg-everywhere. Then for F a function on K, $dF = fdg$ if and only if $f_n dg$ converges in damped variation to dF.*

Proof. Under the hypothesis of Theorem 14 the bounded convergence theorem gives the classical result that f_n converges in measure to f with respect to dg. Since a convergent sequence in $\mathbb{R}$ is bounded, the convergence $f_n \to f$ dg-everywhere yields the existence of v such that $|f_n| \leq v$ dg-everywhere for all n. So by Theorem 12 $f_n dg$ converges in damped variation to fdg. Since limits under convergence in damped variation are unique we get the conclusion of Theorem 14. □

If $dF_n = f_n dg$ with $dg \geq 0$ then f_n is dg-measurable. So for $\sigma = dg \geq 0$ Theorem 14 allows us to adjoin to the six conditions (i), ... , (vi) in Theorem 6 a seventh condition: (vii) $f_n dg$ converges in damped variation to dF.

8. The Cauchy and Harnack extensions.

The Cauchy and Harnack extensions served to evaluate particular types of improper integrals. Such integrals are improper only in the sense that they are excluded by the inadequate definitions of the integral given by Riemann and by Lebesgue. In the wider context of Kurzweil-Henstock integration there are essentially no improper integrals. As Washek Pfeffer says, if an integral can be calculated then it should be integrable. For example, to integrate a summant S over a half-line $[a,\infty)$ map $[a,\infty)$ topologically onto $[a,b)$ with $a < b < \infty$, transforming S into a summant S^* on $[a,b)$. Extend S^* to a summant on $[a,b]$ by setting $S^*(I,t) = 0$ if I has b as an endpoint. The lower and upper integrals of S^* over $[a,b]$ define the corresponding integrals of S over $[a,\infty)$.

With Kurzweil-Henstock the Cauchy and Harnack extensions become convergence theorems. (See Corollaries 7.10 and 7.11 in [9].) We present here general versions of these convergence theorems. In terms of our basic convergence problem these have $T_n = \Delta F_n$ and $T = \Delta F$ with supplementary conditions giving $T_n \to T$ trivially and implying $dF = \sigma$. In the Harnack extension we have $F_n = G_1 + \cdots + G_n$ with some stringent conditions on the functions $G_1, \ldots, G_n, \ldots$. The essence of the Cauchy extension is the implication (iv) $\Rightarrow$ (i) in Theorem 15 for b σ-null.

Theorem 15. *Let σ be a differential on $K = [a,b]$ with b σ-null. Let c be a real number. Then the following are equivalent:*

(i) *σ is integrable on K and its integral over K equals c,*

(ii) *There is a function F on K such that $dF = \sigma$ and $\Delta F(K) = c$,*

(iii) *There is a function F on K such that $F(b-) = F(b)$, $dF = \sigma$ on every cell J in $[a,b)$, and $\Delta F(K) = c$,*

(iv) *σ is integrable on every cell J in $[a,b)$ and $\int_a^x \sigma \to c$ as $x \to b-$.*

Proof. The equivalence of (i) and (ii) is a basic result noted in the introduction. It is essentially Henstock's Lemma [5]. To prove that (ii) implies (iii) note that F given by (ii) is continuous at b because b is σ-null, that is, dF-null. To prove that (iii) implies (iv) we have $\sigma = dF$ on J by (iii). So σ is integrable on $J = [a,x]$ for $a < x < b$ and $\int_J \sigma = \int_J dF = \Delta F(J) = F(x) - F(a)$ which as $x \to b-$ converges to $F(b) - F(a) = \Delta F(K) = c$ by continuity of F at b. This gives (iv).

To complete the proof we prove that (iv) implies (ii). Define the function F on K by

$$F(x) = \int_a^x \sigma \quad \text{for} \quad a \leq x < b, \quad \text{and} \quad F(b) = c. \tag{50}$$

Take a sequence $b_n \longrightarrow b$ with $b_n < b_{n+1} < b$. Then (50) implies $dF = \sigma$ on $[a, b_{n+1}]$. So $1_{[a,b_n]}\,\sigma - dF = 0$ on $[a, b_{n+1}]$, hence on $[a, b]$ since $b_n < b_{n+1} < b$. That is, $[a, b_n]$ is $\sigma - dF$ null for each n in $\mathbb{N}$. The point b is $\sigma - dF$ null since b is σ-null by hypothesis, and dF-null by (50) with continuity of F at b by (iv). So K is the union of countably many $\sigma - dF$ null sets $[a, b_1], \cdots, [a, b_n], \cdots$ and b. Hence K is $\sigma - dF$ null. That is, $\sigma - dF = 0$ on K which gives $\sigma = dF$. Finally, by (50) $\Delta F(K) = F(b) - F(a) = c - 0 = c$. □

Our next two theorems are generalized formulations of the Harnack extension.

Theorem 16. *Let $J_1, \ldots, J_n, \ldots$ be a sequence of nonoverlapping cells in $K = [a, b]$. Let D be the complement in K of $J_1 \cup J_2 \cup \cdots$. Let $G_1, \ldots, G_n, \ldots$ be a sequence of continuous functions on K such that*

$$1_{J_n} dG_n = dG_n \ \text{ on } K \ \text{ for all } n \text{ in } \mathbb{N} \tag{51}$$

and

$$\sum_{n=1}^{\infty} \|G_n\|_\infty < \infty \ \text{ where } \|G\|_\infty = \sup_{t \in K} |G(t)|. \tag{52}$$

Then there is a unique differential ϕ on K such that

$$1_{J_n} \phi = dG_n \ \text{ for all } n \text{ in } \mathbb{N} \tag{53}$$

and

$$D \text{ is } \phi\text{-null.} \tag{54}$$

Moreover, ϕ is integrable. Specifically, $\phi = dF$ for F the continuous function defined by

$$F(x) = \sum_{n=1}^{\infty} G_n(x) \ \text{ for all } x \text{ in } K. \tag{55}$$

Thus,

$$\Delta F(I) = \sum_{n=1}^{\infty} \Delta G_n(I) \ \text{ for every cell } I \text{ in } K \tag{56}$$

and

$$\int_K \phi = \sum_{n=1}^{\infty} \int_{J_n} \phi. \tag{57}$$

Proof. Define the continuous function F by (55) where the series converges absolutely and uniformly by (52). (56) follows directly from (55). By (51) and the continuity of G_n

$$\Delta G_n(I) = \Delta G_n(I \cap J_n) \text{ for every cell } I \text{ in } K. \tag{58}$$

If I is contained in J_n then, since the J_i's are nonoverlapping, (56) and (58) give $\Delta F(I) = \Delta G_n(I)$. So $dF = dG_n$ on J_n. Thus, since F and G_n are continuous, $1_{J_n} dF = 1_{J_n} dG_n$ by (51). That is, (53) holds for $\phi = dF$.

Let $\varepsilon > 0$ be given. To prove (54) we need a gauge δ on K such that

$$\sum_{\mathcal{K}} 1_D |\Delta F| < \varepsilon \tag{59}$$

for every δ-division $\mathcal{K}$ of K. To get such a gauge take N large enough so that by (52)

$$\sum_{n>N} ||G_n||_\infty < \varepsilon/4. \tag{60}$$

Then take δ on K so that

$$\delta(t) < \text{dist}(t, J_1 \cup \cdots \cup J_N) \text{ for all } t \text{ in } D. \tag{61}$$

Such δ exist because D is disjoint from the closed set $J_1 \cup \cdots \cup J_N$. Consider any δ-division $\mathcal{K}$ of K. We contend that (59) holds. Let $(I_1, t_1), \ldots, (I_m, t_m)$ be those members of $\mathcal{K}$ whose tags t_i belong to D. By (61) $I_1 \cup \cdots \cup I_m$ is disjoint from $J_1 \cup \cdots \cup J_N$. So by (56) and (58)

$$\begin{aligned} \sum_{\mathcal{K}} 1_D |\Delta F| &= \sum_{i=1}^{m} |\Delta F(I_i)| = \sum_{i=1}^{m} |\sum_{n>N} \Delta G_n(I_i)| \\ &\leq \sum_{n>N} \sum_{i=1}^{m} |\Delta G_n(I_i)|. \end{aligned} \tag{62}$$

Each J_n cannot contain any of the tags $t_1, \ldots, t_m$ since they belong to D and J_n is disjoint from D. So J_n cannot contain any of the cells $I_1, \ldots, I_m$. Hence, J_n meets at most two of these nonoverlapping cells. So by (58) at most two terms in the sum $\sum_{i=1}^m |\Delta G_n(I_i)|$ are nonzero. Therefore, since each term is bounded by $2||G_n||_\infty$, $\sum_{i=1}^m |\Delta G_n(I_i)| \leq 4||G_n||_\infty$ which with (62) and (60) gives (59). So $1_D dF = 0$ giving (54) for $\phi = dF$. ϕ is tag-null since F is continuous. To get (57) apply (56) with $I = K$ to get $\int_K \phi = \int_K dF = \Delta F(K) = \sum_{n=1}^\infty \Delta G_n(K) = \sum_{n=1}^\infty \int_K dG_n = \sum_{n=1}^\infty \int_K 1_{J_n} \phi = \sum_{n=1}^\infty \int_{J_n} \phi$ by (53) and the tag-nullity of ϕ.

Now consider any differential ϕ on K satisfying (53) and (54). To show that ϕ is unique we must prove $\phi = dF$ for F given by (55). Since both ϕ and dF satisfy (53), $1_{J_n}\phi = dG_n = 1_{J_n} dF$. So $1_{J_n}\ \phi - dF = 0$. That is, J_n is $\phi - dF$ null.

So is D since it is both ϕ-null and dF-null by (54). Thus $K = D \cup J_1 \cup J_2 \cup \cdots$ is $\phi - dF$ null. That is, $\phi - dF = 0$, $\phi = dF$. □

Theorem 17. *Let σ be a tag-null differential on $K = [a, b]$. Let $J_1, \ldots, J_n, \ldots$ be a sequence of nonoverlapping cells in K such that σ is integrable on each J_n, there exists $c < \infty$ such that*

$$\sum_{n=1}^{\infty} \left| \int_{I_n} \sigma \right| < c \text{ for every sequence of cells } I_n \tag{63}$$

$$\text{such that } I_n \text{ is contained in } J_n \text{ for all } n,$$

and $I_D \sigma$ is integrable for D the complement in K of $J_1 \cup J_2 \cup \cdots$. Then σ is integrable on K and

$$\int_K \sigma = \int_K 1_D \sigma + \sum_{n=1}^{\infty} \int_{J_n} \sigma. \tag{64}$$

Proof. Since σ is integrable on J_n and tag-null at the endpoints

$$\int_{J_n} \sigma = \int_K 1_{J_n} \sigma. \tag{65}$$

Thus we can define G_n on K by

$$G_n(x) = \int_a^x 1_{J_n} \sigma. \tag{66}$$

G_n is continuous since σ is tag-null. By Henstock's lemma [5]

$$dG_n = 1_{J_n} \sigma. \tag{67}$$

By (67) and (63) $\sum_{n=1}^{\infty} |\Delta G_n(I_n)| < c$ for all cells I_n contained in J_n. Hence

$$\sum_{n=1}^{\infty} \operatorname{diam} G_n(K) \le c < \infty. \tag{68}$$

$G_n(a) = 0$ by (66). So

$$||G_n||_\infty \le \operatorname{diam} G_n(K). \tag{69}$$

The inequalities (68) and (69) give (52). Apply Theorem 16 to get ϕ satisfying (53) and (54). By (67) $\sigma - 1_D \sigma$ satisfies (53) since D is disjoint from each J_n. It clearly satisfies (54). So by uniqueness $\sigma - 1_D \sigma = \phi$. That is, $\sigma = 1_D \sigma + \phi$. Integrating this we get (64) from (57) since $1_{J_n} \phi = dG_n = 1_{J_n} \sigma$ by (53) and (67). □

References

[1] R. Gordon, *Another look at a convergence theorem for the Henstock integral*, Real Analysis Exchange **15** (1989-90), 724-728.

[2] R. Gordon, *Convergence theorems for the Henstock integral*, Real Analysis Exchange **16** (1990-91), 32-33.

[3] R. Henstock, *Lectures on the theory of integration*, World Scientific (1988).

[4] S. Leader, *What is a differential? A new answer from the generalized Riemann integral*, Amer. Math. Monthly **93** (May 1986), 348-356.

[5] S. Leader, *A concept of differential based on variational equivalence under generalized Riemann integration*, Real Analysis Exchange **12** (1986-87), 144-175.

[6] S. Leader, *1-differentials on 1-cells: a further study*, New Integrals, Lecture Notes in Math. 1419, Springer (1990), 82-96.

[7] S. Leader, *Sequential convergence in lattice groups*, Problems in Analysis, Princeton (1970), 273-290.

[8] P. Y. Lee, *Generalized convergence theorems for Denjoy-Perron integrals*, New Integrals, Lecture Notes in Math. 1419, Springer (1990), 97-109.

[9] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific (1989).

[10] P. Y. Lee, T. S. Chew, *A better convergence theorem for Henstock integrals*, Bull. London Math. Soc. **17** (1985), 557-564.

[11] P. Y. Lee, T. S. Chew, *On convergence theorems for the nonabsolute integrals*, Bull. Austral. Math. Soc. **34** (1986), 133-140.

[12] P. Y. Lee, T. S. Chew, *A short proof of the controlled convergence theorem for Henstock integrals*, Bull. London Math. Soc. **19** (1987), 60-62.

[13] B. Li, *Convergence theorems for the variational integral*, Real Analysis Exchange **15** (1989-90), 324-332.

[14] R. M. McLeod, *The Generalized Riemann Integral*, Carus Math. Monographs **20**, Math. Assoc. Amer. (1980).