Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 87–94

Harvey Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487

Closure of Darboux Graphs

What is the nicest class of functions with the property that the graph of any Darboux function would have the same closure as some member of this class? In 1974, Hugh Miller [6] showed that the graph of any Darboux function $f : I \to I$, where $I = [0, 1]$, has the same closure in I^2 as the graph of some connectivity function $g : I \to I$. Using an analogous transfinite induction argument, he improved this result to obtain that $\bar{f} = \bar{h}$ for some almost continuous function $h : I \to I$ (unpublished). In 1990, at the Seventh Annual Auburn Miniconference on Real Analysis, Ken Kellum asked whether the above results can be generalized so that the function g in Miller's theorem can be chosen to be a connectivity function extendable to a connectivity function from I^2 into I. In this note, we use another technique like in [4] and [3] to show the answer is yes. To illustrate that Miller's result does not generalize to I^2, Kellum gave an example of a Darboux function $f : I^2 \to I^2$ for which $\bar{f} = \bar{h}$ for no almost continuous function $h : I^2 \to I^2$. We end with an equivalence between the uniform closure of the class of Darboux functions and the closure of Darboux graphs.

Let $f : X \to Y$. Then f is <u>Darboux</u> (<u>connectivity</u>) if $f(C)$ (the graph of $f|C$) is connected for every connected subset C of X. We say f is <u>peripherally continuous</u> at x if for each open neighborhood U of x and V of $f(x)$, there is an open neighborhood W of x in U such that $f(\mathrm{bd}(W))\imath V$. We say f is <u>almost continuous</u> if each open neighborhood of the graph of f in $X \times Y$ contains the graph of a continuous function $g : X \to Y$. A connectivity function $f : I \to I$ is said to be <u>extendable</u> if there is a connectivity function $g : I^2 \to I$ such that for all $x \in I$, $g(x, 0) = f(x)$. For functions from I into I, we have:

$$\text{extendable} \Longrightarrow \text{almost continuous} \Longrightarrow \text{connectivity} \Longrightarrow \text{Darboux}$$

where the first arrow is from [8, Cor. 1, Prop. 2] and the second is from [8, Cor., p. 261]. But for functions from I^n into I^m, $n \geq 2$, we have:

$$\text{peripherally continuous} \Longleftrightarrow \text{connectivity} \Longrightarrow \text{almost continuous}$$

where $\Longleftrightarrow$ is from [5, Th. 1] or [9, Cor.] and [8, Th. 4] and $\Longrightarrow$ is from [8, Cor. 1].

Received by the editors Received September 7, 1991

Let K be a simplicial complex, and let a be a point of the underlying polyhedron $|K|$. Then we say a subdivision L of K is obtained by starring at a if L is obtained from K by replacing each simplex Δ of K containing a with all simplexes of the form $a * F$, where F is a face of Δ and $a \notin F$. Here, $a * F$ denotes the join (or cone) of the point a with the set F. A stellar subdivision of K is obtained by starring at points $a_1, a_2, \ldots, a_n \in |K|$ in succession. Figure 1 shows a stellar subdivision S of a 2-simplex Δ^2 resulting from starring at a_1 and then at a_2.

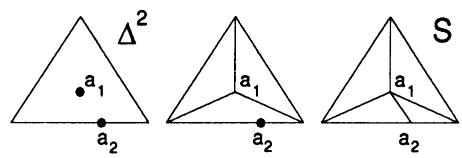

Figure 1:

We let $u_0u_1u_2$ denote the 2-simplex with vertices u_0, u_1, u_2. Other definitions about simplicial complexes needed for the following theorem can be found in [7].

Theorem 1 *For each Darboux function $f : I \longrightarrow I$, there exists an extendable connectivity function $g : I \longrightarrow I$ such that $\bar{f} = \bar{g}$.*

Proof. Let $\Delta^1 = I$. We may identify Δ^1 with $\Delta^1 \times \{0\}$ and let $\Delta^2 = p_0 * \Delta^1$, which is the cone in I^2 with vertex $p_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and base $\Delta^1 \times \{0\}$. Choose a countable dense subset $\{(x_n, f(x_n)) : n = 1, 2, \ldots\}$ of the graph of f where each $x_n \in \text{int}\,(I)$. Let $D = \{x_n : n = 1, 2, \ldots\}$.

Description of L_1: Define $g(0) = f(0)$, $g(1) = f(1), g(p_0) = 0$, and define g to be 0 at the barycenter p of Δ^2. Define $g(x_1) = f(x_1)$ at the point $x_1 \in D$. Let S_1 be the stellar subdivision of Δ^2 obtained by starring at p and then at x_1. Notice that $S_1 = S$ in Figure 1 if $a_1 = p$ and $a_2 = x_1$. Let L_1 denote the 1-skeleton of S_1. For each 1-simplex σ^1 of L_1 which is not contained in Δ^1, extend g linearly on σ^1. For each 2-simplex σ^2 in S_1, the variation of g on $\text{cl}(\text{bd}(\sigma^2) - \Delta^1)$ is ≤ 1.

Description of $L_{m+1} (m \geq 1)$: Suppose we have constructed a simplicial complex S_m so that each 2-simplex of S_m meets Δ^1 and so that the underlying polyhedron $|S_m|$ is the closure of a neighborhood of Δ^1 in Δ^2, and let L_m denote the 1-skeleton of S_m. Suppose that we have defined g on $\text{cl}(|L_m| - \Delta^1) \cup (\Delta^2 - |S_m|)$ so that the following conditions hold:

(1) $x_1, \ldots, x_m$ are vertices of S_m.

(2) g is linear on each 1-simplex of L_m which is not contained in Δ^1.

(3) $g = f$ on the boundary of each 1-simplex of L_m contained in Δ^1.

(4) For each 2-simplex σ^2 in S_m, the variation of g on $\text{cl}(\text{bd}(\sigma^2) - \Delta^1)$ is $\leq \frac{1}{m}$.

(5) g maps $\Delta^2 - |S_m|$ continuously into I.

For the inductive step, we would have to construct S_{m+1} and its 1-skeleton L_{m+1} and define g on $\text{cl}(|L_{m+1}| - \Delta^1) \cup (\Delta^2 - |S_{m+1}|)$ so that conditions (1) - (5) hold for $m + 1$. For simplicity, we instead give a description of just S_2 and L_2 which would be similar to the general case.

Define $g(x_2) = f(x_2)$ at $x_2 \in D$. For argument's sake, we may suppose $x_2 < x_1$. S_1^* denotes the stellar subdivison of S_1 obtained by starring at x_2. Let K be any stellar subdivision of S_1^*, and suppose ρ is a 1-simplex of K with vertices a and b such that $\rho\imath\Delta^1$. If $f(a) \neq f(b)$, then since f is Darboux, there exists a point x in ρ such that $f(x) = \frac{f(a)+f(b)}{2}$, the midpoint of the line segment in I with endpoints $f(a)$ and $f(b)$. But if $f(a) = f(b)$, there may or may not be a point x in $\text{int}(\rho)$ for which $f(x) = \frac{f(a)+f(b)}{2} = f(a) = f(b)$. To remedy this situation, we show how to construct a stellar subdivision K of S_1^* that satisfies the following condition:

(6) If ρ is a 1-simplex of K with vertices a and b such that $\rho\imath\Delta^1$ and $f(a) = f(b)$, then $f(x) = f(a) = f(b)$ for all $x \in \rho$.

Let $\sigma_1 = [0, x_2]$, $\sigma_2 = [x_2, x_1]$, and $\sigma_3 = [x_1, 1]$. For $i = 1, 2, 3$, let $\sigma_i = [c_i, d_i]$. One of the following three cases holds for each σ_i.

<u>Case 1</u>: $f(c_i) \neq f(d_i)$. Then there exists a point x in σ_i such that $f(x) = \frac{f(c_i)+f(d_i)}{2}$.

<u>Case 2</u>: $f(c_i) = f(d_i)$ and f is constant on σ_i. Then every point x in σ_i satisfies $f(x) = \frac{f(c_i)+f(d_i)}{2}$.

<u>Case 3</u>: $f(c_i) = f(d_i)$ and there exists a point w in σ_i such that $f(w) \neq f(c_i) = f(d_i)$. Suppose x_{k_i} is the first point of D in $\text{int}(\sigma_i)$ at which g has not yet been defined such that $f(x_{k_i}) \neq f(c_i) = f(d_i)$. At x_{k_i}, define $g(x_{k_i}) = f(x_{k_i})$. Subdivide S_1^* by starring at x_{k_i}. Then there exist points x in $[c_i, x_{k_i}]$ and x' in $[x_{k_i}, d_i]$ such that $f(x) = \frac{f(c_i)+f(x_{k_i})}{2}$ and $f(x') = \frac{f(x_{k_i})+f(d_i)}{2}$.

Examining which cases hold for σ_1, σ_2, and σ_3, and starring at the point x_{k_i} of D whenever Case 3 occurs, we finally construct a stellar subdivision K of S_1^* that satisfies (6). For example, K would look like Figure 2 if Case 3 occurred for both σ_1 and σ_2 and if Case 1 or 2 occurred for σ_3.

Define a continuous function $\Phi : \Delta^2 \to I$ that is linear on each 2-simplex $\rho^2 = u_0u_1u_2$ of K by the formula $\Phi(l_0u_0+l_1u_1+l_2u_2) = l_0g(u_0)+l_1g(u_1)+l_2g(u_2)$ where $l_0, l_1, l_2 \geq 0$ and $l_0 + l_1 + l_2 = 1$.

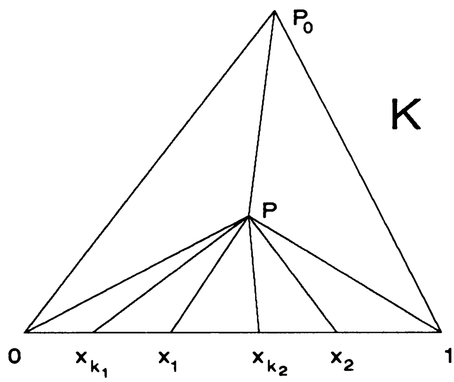

Figure 2:

Let K_1 be the first barycentric subdivision of K. K_1 results from starring at the barycenter of each 2-simplex in K and then at the barycenter of each 1-simplex in K. In other words, K_1 is obtained by drawing the medians of all the triangles belonging to K. We describe another first derived subdivision K^1 of K and a simplicial homeomorphism $\ell_1 : K^1 \to K_1$ as follows.

Let ρ^2 be any 2-simplex of K with a 1-face ρ in Δ^1 with endpoints a and b and midpoint $\hat{\rho}$. In case $f(a) \neq f(b)$, then there exists a point x in ρ such that $f(x)$ is the midpoint $\Phi(\hat{\rho})$ of the line segment from $f(a)$ to $f(b)$. In case $f(a) = f(b)$, then by (6), $f(x) = f(a) = f(b) = \Phi(\hat{\rho})$ for all x in ρ. So in either case, a point x can be chosen in int(ρ) such that $f(x) = \Phi(\hat{\rho})$. A first derived subdivision of each such ρ^2 can be obtained as in Figure 3 by starring first at the barycenter of ρ^2, then at x, and then at the barycenter of each 1-face of ρ^2 that does not lie in Δ^1. For each 2-simplex ρ^2 of K with no 1-face in Δ^1, form the first baycentric subdivision of ρ^2. This resulting first derived subdivision of K is K^1. Define the homeomorphism $\ell_1 : K^1 \to K_1$ this way. If v is a vertex of K^1 and $v =$ some x, where x is as in Figure 3, then $\ell_1(v) = \ell_1(x) = \hat{\rho}$. But if the vertex $v \neq x$, then $\ell_1(v) = v$. Now extend ℓ_1 from the vertices of K^1 so that it linearly maps simplices of K^1 to simplices of K_1.

Define a continuous function $\Phi_1 : K^1 \to I$ by $\Phi_1 = \Phi \circ \ell_1$. It turns out that Φ_1 is linear on each 2-simplex $v_0v_1v_2$ of K^1 and that Φ_1 is given by the formula

$$\Phi_1(l_0v_0 + l_1v_1 + l_2v_2) = \begin{cases} l_0\Phi(v_0) + l_1\Phi(v_1) + l_2\Phi(v_2) \text{ if no } v_i = x \\ l_i\Phi(\hat{\rho}) + \sum_{\substack{k=1 \\ k\neq i}}^{3} l_k\Phi(v_k) \text{ if } v_i = x \end{cases}$$

where $l_0, l_1, l_2 \geq 0$, $l_0 + l_1 + l_2 = 1$, and x is as in Figure 3. Then $\Phi_1(x) =$

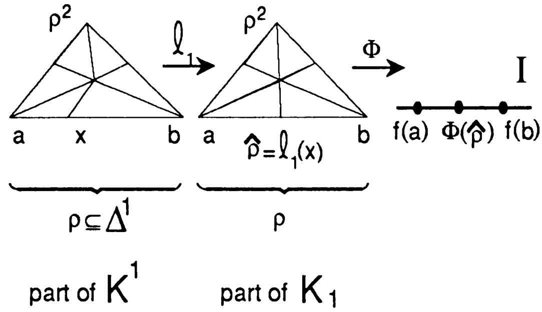

Figure 3:

$\Phi(\ell_1(x)) = \Phi(\hat{\rho}) = f(x)$.

Figure 4 is obtained from Figure 3 the following way. Figure 4 illustrates the second barycentric subdivision K_2 of K (i.e., K_2 is the first barycentric subdivision of K_1) and illustrates the first barycentric subdivision K_1^1 of K^1. It also illustrates another first derived subdivision K^2 of K^1 obtained in a similar way as K^1 was obtained from K. That is, if ρ_i is a 1-simplex of K_1 that is a subset of Δ^1, then x_i' is a point chosen in $\mathrm{int}(\ell_1^{-1}(\rho_i))$ such that $f(x_i') = \Phi(\hat{\rho}_i)$, where $\hat{\rho}_i$ denotes the barycenter of ρ_i. Define a simplicial homeomorphism $\ell_2 : K^2 \to K_1^1$ in a similar way as ℓ_1. Namely, if v is a vertex of K^2 and $v =$ some x_i', then $\ell_2(v) = \ell_2(x_i') = a_i$, which denotes the midpoint of $\ell_1^{-1}(\rho_i)$. But if the vertex $v \neq x_i'$, then $\ell_2(v) = v$. Then extend ℓ_2 so that it maps each 2-simplex of K^2 linearly to a 2-simplex of K_1^1. The continuous function $\Phi_2 : K^2 \to I$ defined by $\Phi_2 = \Phi \circ \ell_1 \circ \ell_2$ is linear on each 2-simplex of K^2 and $\Phi_2(x_i') = \Phi(\ell_1(\ell_2(x_i'))) = \Phi(\ell_1(a_i)) = \Phi(\hat{\rho}_i) = f(x_i')$.

Continuing in this fashion, we obtain for each positive integer n an n^{th} derived subdivision K^n of K, the first barycentric subdivision K_1^{n-1} of K^{n-1}, a simplicial homeomorphism $\ell_n : K^n \to K_1^{n-1}$, and a continuous function $\Phi_n = \Phi \circ \ell_1 \circ \ell_2 \circ \cdots \circ \ell_n : K^n \to I$ which is linear on each 2-simplex of K^n. For some positive integer N, the variation of $\Phi_N : K^N \to I$ on the boundary of each 2-simplex in K^N is $\leq \frac{1}{2}$ because Φ is uniformly continuous on Δ^2 and the mesh of the n^{th} barycentric subdivision K_n of K approaches 0 as $n \to \infty$.

Let $S_2 = \{\sigma : \sigma \text{ is a face of some 2-simplex } \sigma^2 \in K^N \text{ for which } \sigma^2 \cap \Delta^1 \neq \emptyset\}$. $|S_2|$ is the closure of a neighborhood of Δ^1 in Δ^2. Let L_2 be the 1-skeleton of S_2. Observe that $|S_2| \cap |L_1| \mathrm{l} |L_2|$. Define $g = \Phi_N$ on $\mathrm{cl}(|L_2| - \Delta^1) \cup (\Delta^2 - |S_2|)$, and conditions (1) - (5) hold when $m = 2$.

We now suppose that for all $m \geq 1$, conditions (1) - (5) hold and $|S_{m+1}| \cup |L_m|$

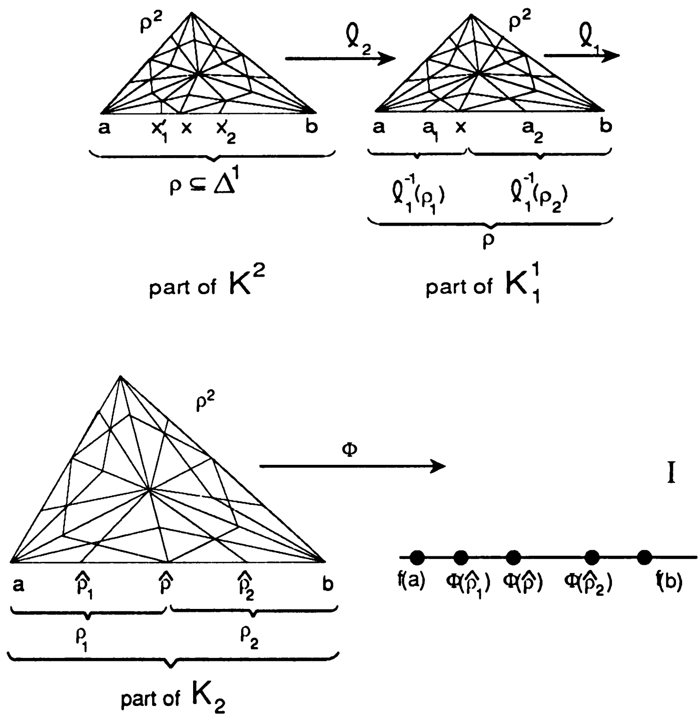

Figure 4:

is a subset of $|L_{m+1}|$. Condition (1) ensures that the mesh of S_m approaches 0 as $m \to \infty$. By construction, g is continuous on $\Delta^2 - \Delta^1$ and peripherally continuous at each point of $\Delta^1 \cap [\bigcup_{m=1}^{\infty} \mathrm{cl}(|L_m| - \Delta^1)]$. Suppose $x_0 \in \Delta^1 - \bigcup_{m=1}^{\infty} \mathrm{cl}(|L_m| - \Delta^1)$. For every m, x_0 lies in the interior (relative to Δ^2) of a 2-simplex s_m of S_m such that as $m \to \infty$, $s_m \to x_0$ and the variation of g on $\mathrm{cl}(\mathrm{bd}(s_m) - \Delta^1)$ approaches 0. If we choose $y_m \in \mathrm{bd}(s_m \cap \Delta^1)$, then $y_m \to x_0$. Define $g(x_0)$ to be a cluster point of $f(y_1), f(y_2), \ldots$. Then $g : \Delta^2 \to I$ is peripherally continuous at x_0 and therefore a connectivity function. The graphs of f and the extendable function $g|\Delta^1$ have the same closure because $g = f$ on the set $\Delta^1 \cap [\bigcup_{m=1}^{\infty} \mathrm{cl}(|L_m| - \Delta^1)]$ containing D and because of the above way $g(x_0)$ is defined at the other points x_0 of Δ^1.

Question. In Theorem 1, can g be chosen to be measurable whenever f is?

A real-valued function $f : I \to R$ is defined to be in the class $\mathcal{U}$ if for every interval $[a,b]\imath I$ and every subset A of $[a,b]$ with less than c-many points, the set $f([a,b]-A)$ is dense in the closed interval with endpoints $f(a)$ and $f(b)$. A function $f : I \to R$ is in the uniform closure of the class $\mathcal{D}$ of Darboux functions if it is the uniform limit of a sequence of Darboux functions $f_n : I \to R$. That is, f is a closure point of $\mathcal{D}$ in the space of all functions $I \to R$ with the metric ϱ of uniform convergence described in [1] this way: For functions $f,g : I \to R$, let $\sigma(f,g) = \sup\{|f(x)-g(x)| : x \in I\}$.

$$\text{Define } \varrho(f,g) = \begin{cases} 1 \text{ if } \sigma(f,g) = \infty \\ \frac{\sigma(f,g)}{1+\sigma(f,g)} \text{ otherwise.} \end{cases}$$

According to [2], the class $\mathcal{U}$ is the uniform closure of $\mathcal{D}$. The uniform closure $\bar{\mathcal{D}}$ and closure of a Darboux function turn out to be related in the following sense:

Theorem 2 *Let $f : I \to I$. Then $f \in \bar{\mathcal{D}}$ if and only if*

(a) f is bilaterally c-dense in itself and unilaterally at the endpoints, and

(b) there exists a function $g : I \to I$ such that $g \in \mathcal{D}$ and $\bar{f} = \bar{g}$.

Proof. Suppose $f \in \bar{\mathcal{D}} = \mathcal{U}$. This implies that (a) holds [2]. Miller gave an argument that $f \in \bar{\mathcal{D}} \Rightarrow$ (b) like this: For each $x \in I$, $\bar{f} \cap (\{x\} \times I)$ is connected, which along with (a) is enough to conclude as in Theorem 1 of [6] that there exists a connectivity function g such that $\bar{f} = \bar{g}$. So (b) holds.

Now suppose (a) and (b) hold. Let $[a,b]\imath I$, and let A be a subset of $[a,b]$ with less than c-many points. We may as well suppose $f(a) < w < f(b)$. Since f is bilaterally dense in itself and $\bar{f} = \bar{g}$ for some Darboux function $g : I \to I$, there exist $c,d \in [a,b]$ such that $g(c), g(d) \in (f(a), f(b))$ and $g(c) < w < g(d)$. Given $\varepsilon > 0$, there exists a point (z,w) of g and therefore of $\bar{f}$ that belongs to $(a,b)\times(w-\varepsilon, w+\varepsilon)$. Therefore some point $(x_0, f(x_0))$ lies in $(a,b)\times(w-\varepsilon,w+\varepsilon)$. Because f is c-dense in itself, there is a point $u \in (a,b)-A$ such that $(u,f(u)) \in (a,b)\times(w-\varepsilon, w+\varepsilon)$. Then $f(u) \in (w-\varepsilon, w+\varepsilon)$ implies $f([a,b]-A)$ is dense in $[f(a), f(b)]$. That is, $f \in \mathcal{U} = \bar{\mathcal{D}}$.

References

[1] A. M. Bruckner and J. G. Ceder, *On jumping functions by connected sets*, Czechoslovak Math. J. 22(1972), 435-448.

[2] A. M. Bruckner, J. G. Ceder, and M. Weiss, *Uniform limits of Darboux functions*, Colloq. Math. 15(1966), 65-77.

[3] R. G. Gibson, H. Rosen, and F. Roush, *Connectivity functions* $I^n \to I$ *dense in* $I^n \times I$, Real Analysis Exchange 14(1989), 99-103.

[4] R. G. Gibson and F. Roush, *Connectivity functions defined on* I^n, Colloq. Math. 55(1988), 41-44.

[5] M. R. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. 17(1966), 175-177.

[6] M. H. Miller, *Equivalence of certain discontinuous functions under closure*, Proc. Amer. Math. Soc. 54(1976), 384-388.

[7] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, Springer–Verlag, Berlin, 1972.

[8] J. R. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47(1959), 249-263.

[9] G. T. Whyburn, *Connectivity of peripherally continuous functions*, Proc. Nat. Acad. Sci. U.S.A. 55(1966), 1040-1041.