Real Analysis Exchange
Vol. 18(1), 1992/93, pp. 70–81

G. Petruska, Department of Mathematics, University of the Witwatersrand, Johannesburg 2050, South Africa.

On the Relative Grid Dimension of Continuous Functions

The n^{th} grid G_N on the unit square $S = [0,1] \times [0,1]$ is the se t of elementary closed squares of the regular $n \times n$ subdivision of S. For any $E \subset S$, $E \neq \phi$ let $N(E, n)$ denote the number of elements of G_n which meet E. For a given subsequence of natural numbers $\nu(n)$ $(n = 1, 2, \ldots)$ the grid dimension $\alpha_\nu(E)$ of a set E relative to the sequence ν is defined by

$$\begin{aligned} \alpha_\nu(E) &= \inf\{\alpha : \limsup_{n\to\infty} \frac{N(E,\nu(n))}{\nu(n)^\alpha} < \infty\} \\ &= \sup\{\alpha : \limsup_{n\to\infty} \frac{N(E,\nu(n))}{\nu(n)^\alpha} = \infty\}, \end{aligned}$$

or equivalently

$$\alpha_\nu(E) = \limsup_{n\to\infty} \frac{\log N(E,\nu(n))}{\log \nu(n)}. \qquad (*)$$

For $\nu(n) = n$ $(n = 1, \ldots)$ we put $\alpha_\nu(E) = \alpha(E)$ and this number is called the grid dimension. Obviously, $0 \leq \alpha_\nu(E) \leq \alpha(E) \leq 2$ for any ν and $E \neq \phi$. In this paper we study the growth conditions on ν implying $\alpha_\nu(E) = \alpha(E)$ for any E at one hand, and on the other, with special attention to the case when $E = \Gamma_f$, the graph of a continuous function f.

The exact value of the rarefaction index

$$\tau = \inf\{t : \text{ the grid dimension was known in the year } t\}$$

is not known, but certainly $\tau \leq 1928$. This dimension, perhaps the first time was used by Bouligand in [BO], 1928 (see also [MA] for references). As it turns out, it has been reintroduced by several authors, each giving to it a new name (see [FA], p. 38), and as a result, this single concept now enjoys such a long list of titles that seeing it, even a Spanish Grandee could turn green with envy. It is

Received by the editors September 4, 1991

mostly known as box dimension (more precisely upper box counting dimension) and the terms grid and relative grid dimension were introduced in [HP].

It has been observed in [HP] (see also a remark in [FA], p. 41], that $\alpha_\nu = \alpha$, if $\nu(n+1) \leq c\nu(n)$. This allows us to compute the grid dimension on special grids, for instance refining the grid by a repeated halving procedure (the case $\nu(n) = 2^n$). We may as well say, that taking only $\log n$ of the first n grids, the dimension will be the same as using the full sequence. It is a natural question to ask, that how many of the grids actually needed to make sure that the relative dimension equals the dimension for any set, or for any continuous graph. The answer is "more than $\log\log n$" in both cases, that is condition (3) below is the necessary and sufficient condition to ensure $\alpha_\nu(E) = \alpha(E)$ (see Theorems 1 and 3). The main motivation of this paper (as well as that of [HP]) was to find badly behaving continuous graphs for which the limsup in (*) is <u>not</u> a limit. This is why we formulate the necessity of (3) in a stronger form for graphs and separately under Theorem 3. Grids on **R** have already been studied by Tricot in 1980 (see [TR]) and Theorem 1 was obtained for one dimensional grids. Though the higher dimensional case is quite a staightforward generalization, we present here a simple proof, partly for the reader's convenience ([TR] is hardly available), partly because the method is needed for subsequent applications. Theorem 3 holds true for the 3 and higher dimensional cases as well, though the construction needs more care because of the more complicated nature of surfaces.

Lemma 1. *Let $1 \leq k \leq n$ be arbitrary integers, $E \subset S$. Then*

$$N(E,k) \leq 4N(E,n) \tag{1}$$

$$N(E,n) \leq (\frac{n}{k}+2)^2 N(E,k). \tag{2}$$

Proof. Let $Q \in G_n$ and $k < n$. If Q does not contain a vertex of G_k, then Q is either covered by the interior of a square from G_k, or it is covered by the interior of the union of two adjacent elements of G_k. If Q contains a vertex v of the grid G_k, then there are at most four elements of G_k joining at v. Taking union of these squares, the interior of this union contains Q. Therefore in each case $N(Q,k) \leq 4$ and hence (1) follows. Turning to (2) we observe, that an interval of length $1/k$ can intersect at most $\frac{n}{k}+2$ non-overlapping intervals of length $1/n$ and hence an element of G_k cannot meet more than $(\frac{n}{k}+2)^2$ elements of G_n, that is $N(Q,n) \leq (\frac{n}{k}+2)^2$. Therefore

$$N(E,n) \leq \sum_{\substack{Q\in G_k \\ Q\cap E\neq\phi}} N(Q,n) \leq (\frac{n}{k}+2)^2 N(E,k).$$

Theorem 1. *Let $\nu(n)$ $(n = 1, 2, \ldots)$ be a subsequence of the natural numbers. If*

$$\lim_{n\to\infty} \frac{\log \nu(n+1)}{\log \nu(n)} = 1, \tag{3}$$

then $\alpha_\nu(E) = \alpha(E)$ for any $E \in S$.

Proof. For any k we can choose n with $\nu(n) \le k \le \nu(n+1)$, then by (1) we have $N(E, k) \le 4N(E, \nu(n+1))$, and hence

$$\frac{\log N(E,k)}{\log k} \le \frac{\log 4}{\log \nu(n)} + \frac{\log N(E, \nu(n+1))}{\log \nu(n+1)} \cdot \frac{\log \nu(n+1)}{\log \nu(n)}.$$

Taking the limsup on both sides we obtain $\alpha_\nu(E) \ge \alpha(E)$ and the proof is complete.

Corollary 1. *However slowly the increasing sequence $\rho(n)$ tends to infinity, for $a > 1, b > 1$ and*

$$\nu(n) = a^{b^{n/\rho(n)}}$$

we have $\alpha_\nu(E) = \alpha(E)$.

Proof.

$$\log\log \nu(n+1) - \log\log \nu(n)$$

$$= \log b \left(\frac{n+1}{\rho(n+1)} - \frac{n}{\rho(n)} \right) \le \log b \left(\frac{n+1}{\rho(n)} - \frac{n}{\rho(n)} \right) = \frac{\log b}{\rho(n)} \to 0.$$

Theorem 2. *Let two sequences $\mu(n)$, $\nu(n)$ be given. If*

$$\lim_{n\to\infty} \frac{\log \mu(n)}{\log \nu(n)} = 1, \tag{4}$$

then $\alpha_\nu(E) = \alpha_\mu(E) (E \subset S)$.

Proof. Let n run through the integers satisfying $\nu(n) \le \mu(n)$. Taking logarithms in (1) we obtain

$$\frac{\log N(E, \nu(n))}{\log \nu(n)} \le \frac{\log N(E, \mu(n))}{\log \mu(n)} \frac{\log \mu(n)}{\log \nu(n)} + o(1),$$

and similarly, by (2) and (4) we get

$$\frac{\log N(E, \mu(n))}{\log \mu(n)} \frac{\log \mu(n)}{\log \nu(n)} \le \frac{\log N(E, \nu(n))}{\log \nu(n)} + o(1).$$

Referring to (4) again,

$$\limsup_{\substack{n\to\infty \\ \nu(n)\le\mu(n)}} \frac{\log N(E,\nu(n))}{\log \nu(n)} = \limsup_{\substack{n\to\infty \\ \nu(n)\le\mu(n)}} \frac{\log N(E,\mu(n))}{\log \mu(n)}.$$

By symmetry we obtain the same result on the sequence $\nu(n) \ge \mu(n)$ and hence $\alpha_\nu(E) = \alpha_\mu(E)$ as stated.

The next statement easily follows from Theorem 2.

Corollary 2. *Let a subsequence $\kappa(n) = \nu(k_n)$ of $\nu(n)$ be given. If for any arbitrary sequence j_n of integers satisfying $k_n \le j_n \le k_{n+1} - 1$ we have*

$$\lim_{n\to\infty} \frac{\log \nu(k_n)}{\log \nu(j_n)} = 1,$$

then $\alpha_\kappa(E) = \alpha_\nu(E)$ for any $E \subset S$. (Notice that

$$\lim_{n\to\infty} \frac{\log \nu(k_n)}{\log \nu(k_{n+1}-1)} = 1$$

is equivalent to our assumption.)

Proof. Let E be given and let a sequence p_n be selected such a way that

$$\alpha_\nu(E) = \lim_{n\to\infty} \frac{\log N(E,\nu(p_n))}{\log \nu(p_n)}.$$

Since we have a limit here, we may further rarify p_n, and hence suppose without loss of generality, that $k_{\ell(n)} \le p_n \le k_{\ell(n)+1} - 1$ $(n = 1,2,\ldots)$ for a suitable sequence $\ell(n)$. By our assumption and Theorem 2 we have $\alpha_{\kappa\circ\ell}(E) = \alpha_\nu(E)$, that is $\alpha_\kappa(E) \ge \alpha_\nu(E)$. On the other hand, $\alpha_\kappa(E) \le \alpha_\nu(E)$ is obvious, and hence the statement.

Lemma 2. *For every sequence $\nu(n)$ and for every natural number N there exists another sequence $\mu(n)$ such that*

(i) $\mu(n-1) \mid \mu(n)$, $\mu(n) \ge 2\mu(n-1)$ $(n = 2,3,\ldots)$;

(ii) *every $\mu(n)$ is a full N^{th} power:* $\mu(n)^{1/N} \in \mathbb{Z}$ $(n = 1,2,\ldots)$;

(iii)

$$\limsup_{n\to\infty} \frac{\log \nu(n+1)}{\log \nu(n)} = \limsup_{n\to\infty} \frac{\log \mu(n+1)}{\log \mu(n)};$$

(iv) $\alpha_\nu(E) = \alpha_\mu(E)$ *for every* $E \in S$.

Proof. Let $\mu(1) = 1$. Suppose that $\mu(n)$ has been defined and for $1 \le k \le n$ properties (i) and (ii) hold true. We choose now the least $k = k_n$ such that $\nu(k) \ge 2^N \mu(n)$, then put

$$\mu(n+1) = \left[\left(\frac{\nu(k_n)}{\mu(n)}\right)^{1/N}\right]^N \mu(n),$$

where $[x]$ stands for the integer part of x. Properties (i), (ii) are obvious by induction. Also,

$$\begin{aligned} \nu(k_n) &\ge \mu(n+1) \ge \left(\left(\left(\frac{\nu(k_n)}{\mu(n)}\right)^{1/N} - 1\right)\mu(n)^{1/N}\right)^N \\ &= \nu(k_n)\left(1 - \left(\frac{\mu(n)}{\nu(k_n)}\right)^{1/N}\right)^N \ge \frac{\nu(k_n)}{2^N}, \end{aligned}$$

and hence

$$\frac{\log \nu(k_n)}{\log \mu(n)} \to 1. \tag{5}$$

Thus putting $\kappa(n) = \nu(k_n)$, Theorem 2 implies $\alpha_\kappa(E) = \alpha_\mu(E)$. If $k_n \le j \le k_{n+1} - 1$ then by the minimum choice of k_{n+1} we obtain

$$\nu(k_n) \le \nu(j) \le \nu(k_{n+1} - 1) < 2^N \mu(n+1) \le 2^N \nu(k_n), \tag{6}$$

hence

$$1 \le \frac{\log \nu(j)}{\log \nu(k_n)} \le 1 + \frac{N \log 2}{\log \nu(k_n)},$$

and Corollary 2 implies $\alpha_\kappa(E) = \alpha_\nu(E)$, that is $\alpha_\mu(E) = \alpha_\nu(E)$. Making use of (6) again we also obtain

$$\begin{aligned} \frac{\log \nu(j+1)}{\log \nu(j)} &\le \frac{\log \nu(k_{n+1})}{\log \nu(k_n)} = \frac{\log \nu(k_{n+1})}{\log \nu(k_{n+1} - 1)} \cdot \frac{\log \nu(k_{n+1} - 1)}{\log \nu(k_n)} \\ &\le \frac{\log \nu(k_{n+1})}{\log \nu(k_{n+1} - 1)}\left(1 + \frac{N \log 2}{\log \nu(k_n)}\right), \end{aligned}$$

thus

$$\limsup_{n\to\infty} \frac{\log \nu(n+1)}{\log \nu(n)} = \limsup_{n\to\infty} \frac{\log \nu(k_{n+1})}{\log \nu(k_n)},$$

and now (iii) follows by (5), making our proof complete.

Our next theorem shows that Theorem 1 is the best possible, even for continuous graphs.

Theorem 3. *Suppose that the sequence $\nu(n)$ satisfies*

$$r = \limsup_{n\to\infty} \frac{\log \nu(n+1)}{\log \nu(n)} > 1. \tag{7}$$

Then for every $\varepsilon > 0$ there exists a continuous graph Γ_f such that

$$\alpha(\Gamma_f) \geq \alpha_\nu(\Gamma_f) + \frac{\sqrt{r}-1}{\sqrt{r}+1} - \varepsilon. \tag{8}$$

Remarks.

1. With some extra pain, ε could be eliminated from (8). But we do not know anyway, what would be that best lower estimate in (8) (see the Problem below).

2. The extreme case $r = \infty$, that is $2 = \alpha(\Gamma_f) = \alpha_\nu(\Gamma_f) + 1$ was stated and proved in [HP], Theorem 7, under the stronger condition that the limit (not just the limsup) in (7) was infinity.

3. It has been pointed out by the referee that, a shorter and simpler construction gives even a sharper result in Theorem 3, if any subset $E \subset S$ can be considered. He showed that assuming (7), there exists $E \subset S$ such that
$$\alpha(E) = \alpha_\nu(E) + \frac{r-1}{r+1}. \tag{9}$$
His construction yields a set $E = C \times C$, where C is a suitable Cantor type set on the line (this method readily generalizes for higher dimensions). Unfortunately I was not able to transform E into a perfect subset of a continuous graph. A tilting transformation like $\tau : (x, y) \mapsto (x + y, y)$ looks promising, because it does not change the relative grid dimension. But to ensure that $\tau(E)$ is a graph above its projection, we need also a strong independence w.r.t translations $|C \cap (C + x)| \leq 1$ for any $x \in C$, which looks hard to control together with the dimensional requirements. Thus the following problem remains open.

Problem. Is (9) available for graphs? More generally, assuming (7) for ν, determine $\inf_\nu \sup_{\Gamma_f} (\alpha(\Gamma_f) - \alpha_\nu(\Gamma_f))$.

We shall use the following notations and definitions:

- $[x]$ denotes the integer part of x;
- in what follows, c denotes a quantity which may depend on n, it may vary from side to side of equations an inequalities or from formula to formula, but $0 < a \le c \le b < \infty$ must hold, where a and b are constants;
- $|H|$ denotes the cardinality of the set H;
- for $C \in G_n$ we put $H(C) = \cup\{Q : Q \in C\}$;
- pr_x and pr_y denote the projections onto the x and y axis, respectively;
- $C \in G_n$ (and the corresponding $H(C)$) is said to be a column, if $pr_x(Q_1) = pr_x(Q_2)$ for any $Q_1, Q_2 \in C$ and $pr_y(H(C))$ is an interval; the length of this interval is denoted by $||C||$ or $||H(C)||$;
- $C \in G_n$ (and the corresponding $H(C)$) is said to be a graph, if $H(C)$ is connected, $pr + x(H(C)) = [0,1]$, and for any $Q \in C$ the set $C' = \{Q' \in C : pr_xQ' = pr_xQ\}$ is a column, for any graph C we put $||C|| = ||H(C)|| = \max\{||C'|| : C' \subset C,\ C' \text{ a column}\}$.

The two propositions below are to be verified easily, and the details are left to the reader (see also [HP]).

Proposition 1. *If C_k is a sequence of graphs such that $H(C_{k+1}) \subset H(C_k)$ and $||C_k|| \to 0$, then $\cap H(C_k)$ is the graph of a continuous function.*

Proposition 2. *Let a graph $C \subset G_N$ be given. Let $M = kN$, $k \ge 2$ and $1 \le \lambda \le k$. Then there exists a graph $D \subset G_M$ (called the λ-refinement) such that*

(i) $H(D) \subset H(C)$,

(ii) *denoting $D_Q = \{Q' : Q' \in D,\ Q' \subset Q\}$, $k\lambda \le |D_Q| \le k(\lambda + 1)$ holds for every $Q \in C$ (hence $k\lambda|C| \le |D| \le k(\lambda + 1)|C|$ holds as well);*

(iii) *for $\lambda \le [k/2]$ we may prescribe $||D|| \le \frac{3}{4}||C||$.*

The special cases $\lambda = 1$ and $\lambda = [k, 2]$ (refinements of 1^{st} and 2^{nd} kind) are illustrated on the figure, showing how the columns of the finer graph are selected and connected to each other, and also that in the second case the norm is about the half of the previous one. In the general case we choose λ new columns in each of the former columns, and in each old square we need at most k connecting

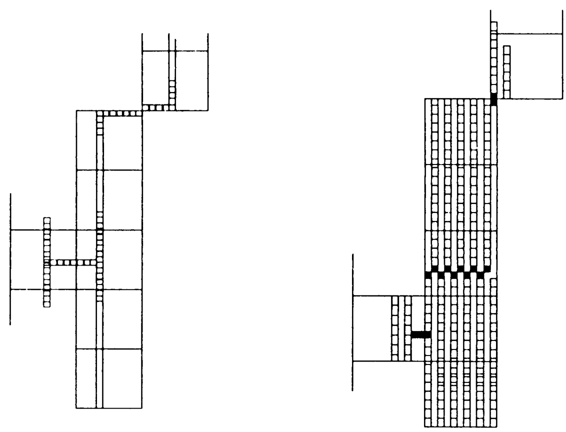

Figure 1: Refinement of the first kind, $k = \frac{M}{N} = 8$ and refinement of the second kind, $k = \frac{M}{N} = 9$

smaller squares between them, which explains $k(\lambda + 1)$ in the upper estimate. The choice of λ controls the change of the dimension of the underlying graphs.

Proof of Theorem 3. Let us suppose now, that $\nu(n)$ satisfies (7) and choose $1 < u < r$. Let A, B, N be natural numbers, $A + B = N$ and let $p = \frac{A}{N}$, $q = \frac{B}{N}$. Referring to Lemma 2 we can replace $\nu(n)$ by another sequence satisfying (i), (ii), (iii) and in particular, preserving the relative grid dimension. Thus we can suppose without loss of generality, that $\nu(n)$ itself satisfies (i), (ii), (iii) of Lemma 2. Applying (7) we can select a subsequence k_n such that for every n, $\log \nu(k_n + 1) > u \log \nu(k_n)$. Let

$$t(n) = \nu(k_n)^p \nu(k_n + 1)^q \ (n = 1, 2, \ldots).$$

Since $\mu(n), \mu(n+1)$ are perfect N^{th} powers, $t(n)$ is an integer and $\nu(k_n)|t(n)$, $t(n)|\nu(k_n + 1)$. We define two sequences of graphs $L(n) \subset G_{\nu(n)}$, $T(n) \subset G_{t(n)}$ by induction as follows. Let $L(1) = G_{\nu(1)}$, and suppose we have defined the graph $L(n) \subset G_{\nu(n)}$ for some n. If $n = k_j$, then we apply Proposition 2 using a refinement of second kind ($\lambda = [k/2]$) on $L(k_j)$ and we get a finer graph $T(j) \subset G_{t(j)}$; next we continue by a refinement of first kind on $T(j)$ and obtain the finer graph $L(k_j + 1) \subset G_{\nu(k_j+1)}$. If, on the other hand, $k_j + 1 \leq n < k_{j+1}$ for some j, then again referring to Proposition 2, we apply a λ refinement to obtain $L(n+1) \subset G_{\nu(n+1)}$ such a way that the ratio $\frac{\log |L(n)|}{\log \nu(n)}$ should change as little as possible in the block $k_j + 1 \leq n \leq k_{j+1}$. Accordingly, λ is chosen as follows: if

$$\frac{\log|L(n)|}{\log\nu(n)} \geq \frac{\log|L(k_j+1)|}{\log\nu(k_j+1)},$$

then we choose the largest possible λ for which

$$\frac{\log|L(n+1)|}{\log\nu(n+1)} \leq \frac{\log|L(k_j+1)|}{\log\nu(k_j+1)}$$

holds (put $\lambda = 1$, if the opposite inequality holds for any λ). If

$$\frac{\log|L(n)|}{\log\nu(n)} < \frac{\log|L(k_j+1)|}{\log\nu(k_j+1)},$$

then λ is the least possible to validate

$$\frac{\log|L(n+1)|}{\log\nu(n+1)} \geq \frac{\log|L(k_j+1)|}{\log\nu(k_j+1)}$$

(put $\lambda = k$, if the opposite inequality holds for any λ), By (ii) of Proposition 2 we have

$$\lambda\frac{\nu(n+1)}{\nu(n)}|L(n)| \leq |L(n+1)| \leq (\lambda+1)\frac{\nu(n+1)}{\nu(n)}|L(n)|, \tag{10}$$

that is, the upper estimate for the λ refinement is the same as the lower estimate for $\lambda+1$ refinement. Taking the logarithm and dividing by $\log\nu(n+1)$, (10) shows that $\frac{\log|L(n+1)|}{\log\nu(n+1)}$ is located in an interval of length

$$\frac{\log(1+\frac{1}{\lambda})}{\log\nu(n+1)} < \frac{1}{\log\nu(n+1)}.$$

Therefore, by the optimal choice of λ as described above, we obtain

$$\left|\frac{\log|L(n)|}{\log\nu(n)} - \frac{\log|L(k_j+1)|}{\log\nu(k_j+1)}\right| \leq \frac{1}{\log\nu(k_j+1)} \quad (k_j < n \leq k_{j+1}). \tag{11}$$

Since $\|L(n+1)\| \leq \|L(n)\|$ for any n and $\|L(k_n+1)\| \leq \|T(n)\| \leq \frac{3}{4}\|L(k_n)\|$, by Proposition 1

$$\Gamma_f = \bigcap_{n=1}^{\infty} H(L(n))$$

is indeed the graph of a continuous function. There remains to show, that the dimensions of Γ_f satisfy the estimate as stated. By the definition of Γ_f

the estimates $|L(n)| \le N(\Gamma_f, \nu(n)) \le 9|L(n)|$ and $|T(n)| \le N(\Gamma_f, t(n))$, are obvious, thus

$$\alpha_\nu(\Gamma_f) = \limsup \frac{\log|L(n)|}{\log \nu(n)}, \text{ and } \alpha(\Gamma_f) \ge \alpha_t(\Gamma_f) \ge \limsup \frac{\log|T(n)|}{\log t(n)}.$$

By (11)

$$\limsup_j \frac{\log|L(k_j+1)|}{\log \nu(k_j+1)} = \limsup_n \frac{\log|L(n)|}{\log \nu(n)}.$$

Therefore

$$\alpha_\nu(\Gamma_f) = \limsup_{n\to\infty} \frac{\log|L(k_n+1)|}{\log \nu(k_n+1)}. \tag{12}$$

Finally we have to compute $|L(k_n+1)|$ and $|T(n)|$. We have

$$\begin{aligned} |L(k_n+1)| &= c\frac{\nu(k_n+1)}{t(n)}|T(n)| = c\frac{\nu(k_n+1)}{t(n)} \cdot \frac{t(n)^2}{\nu(k_n)^2}|L(k_n)| \\ &= c\left(\frac{\nu(k_n+1)}{\nu(k_n)}\right)^{1+q} |L(k_n)|. \end{aligned}$$

Here $c \le 4$. Taking logarithms and introducing the notation

$$d_n = \left| \frac{\log|L(k_n+1)|}{\log \nu(k_n+1)} - (1+q) \right|$$

we obtain by (11)

$$d_n = \left| \frac{\log c}{\log \nu(k_n+1)} + \frac{\log \nu(k_n)}{\log \nu(k_n+1)} \cdot \left(\frac{\log|L(k_n)|}{\log \nu(k_n)} - (q+1) \right) \right| \le \frac{1}{u} d_{n-1} + \frac{3}{\log \nu(k_{n-1})},$$

that is, applying this for $n = 2, \dots, n+1$ we get

$$d_{n+1} \le \frac{d_1}{u^n} + \sum_{j=1}^{n} \frac{\gamma_j}{u^{n-j}},$$

where $\gamma_j = \frac{3}{\log \nu(k_j)} \to 0$. Therefore $d_n \to 0$ and by (12) we get

$$\alpha_\nu(\Gamma_f) = 1 + q, \tag{13}$$

moreover, making use of (11) once again

$$\lim_{n\to\infty} \frac{\log|L(n)|}{\log \nu(n)} = 1 + q \tag{14}$$

holds true. Taking the logarithms in

$$|T(n)| = c\frac{t(n)^2}{\nu(k_n)^2}|L(k_n)|,$$

(14) and the definition of $t(n)$ imply

$$\log|T(n)| \geq \log c + 2\log t(n) - 2\log\nu(k_n) + (1+q-\eta)\log\nu(k_n)$$

$$= \log c + \log t(n) - \eta\log\nu(k_n) + q\log\nu(k_n+1) \qquad (15)$$

for arbitrary $\eta > 0$ and large n. By the definition of the sequence $\{k_n\}$, there holds the estimate $\log\nu(k_n+1) \geq u\log\nu(k_n)$, or equivalently, as shown by a short computation

$$\log\nu(k_n+1) \geq \frac{u}{p+uq}\log t(n).$$

Putting this in (15) we obtain

$$\log|T(n)| \geq \log c + \frac{p+2uq}{p+uq}\log t(n) - \eta\log\nu(k_n),$$

and since η was arbitrary, $\alpha_t(\Gamma_f) \geq \frac{p+2uq}{p+uq}$. This and (13) imply

$$\alpha(\Gamma_f) - \alpha_\nu(\Gamma_f) \geq \alpha_t(\Gamma_f) - \alpha_\nu(\Gamma_f) \geq \frac{p+2uq}{p+uq} - (1+q) = \frac{(u-1)pq}{p+uq}.$$

By elementary calculation, the function $F(x,y) = \frac{xy(u-1)}{x+yu}$ $(x+y=1)$ attains its maximum for $y = \frac{1}{\sqrt{u}+1}$ and here

$$F_{\max} = \frac{\sqrt{u}-1}{\sqrt{u}+1}.$$

Therefore, if we choose the rational number $q = \frac{B}{N}$ close enough to $\frac{1}{\sqrt{u}+1}$, we obtain

$$\alpha(\Gamma_f) - \alpha_\nu(\Gamma_f) \geq \frac{\sqrt{u}-1}{\sqrt{u}+1} - \varepsilon.$$

Since $\varepsilon > 0$ and $u < r$ are arbitrary, the proof is complete.

Acknowledgement. The author is indebted to the referee for his valuable comments and suggestions.

References

[BO] G. Bouligand, *Ensembles impropres et nombre dimensional*, Bull. des Sciences Math. 52 (1928).

[FA] K.J. Falconer, *Fractal Geometry*, J. Wiley & Sons, 1990.

[MA] B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[HP] P.D. Humke, G. Petruska, *The packing dimension of a typical continuous function is 2*, Real Analysis Exchange, 1989, 345-358.

[TR] C. Tricot, *Sur la classification des ensembles boreliens de mesure de Lebesgue nulle*, Thesis, Geneve, 1980.