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Copson Type Inequalities with Weighted Means

1. Introduction

Copson [C] proved the following inequalities:

Theorem A *Let $p > 1$, $a_n > 0$, $q_n > 0$, $Q_n := q_1 + \ldots + q_n$ for $n = 1, 2, \ldots$, and $\sum_{n=1}^{\infty} q_n a_n^p < \infty$. Then*

$$\sum_{n=1}^{\infty} q_n \left[\frac{1}{Q_n} \sum_{k=1}^{n} q_k a_k \right]^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} q_n a_n^p; \tag{1.1}$$

$$\sum_{n=1}^{\infty} q_n \left[\sum_{k=n}^{\infty} \frac{q_k a_k}{Q_k} \right]^p \leq p^p \sum_{n=1}^{\infty} q_n a_n^p. \tag{1.2}$$

In the special case $q_k = 1$, $Q_n = n$, Hardy's inequality is obtained [HLP, p.239]. The following theorems give a pair of related inequalities recently obtained by Mohapatra et. al. [MRV].

Theorem B *Let $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $q_n > 0$, $Q_n := q_1 + \cdots + q_n$, $a_n \geq 0$. Write $\vec{\Delta}\, U_n = U_n - U_{n+1}$. If $nq_n \leq AQ_n$ and $n|\vec{\Delta}\, {q_n}^{1/p'}| \leq B_p {q_n}^{1/p'}$ for some constants A and B_p with $n = 1, 2, \ldots$, then for each $N \geq 1$*

$$\sum_{n=1}^{N} q_n \left[\frac{1}{Q_n} \sum_{k=1}^{n} q_k a_k \right]^p \leq K(p) \sum_{n=1}^{N} \left[\frac{1}{n} \sum_{k=1}^{n} {q_k}^{1/p} a_k \right]^p \tag{1.3}$$

where $K(p) \leq \left[A + \frac{p^2}{p-1} B_p \right]^p$.

Theorem C *With notation as in Theorem B, suppose $nq_n \leq AQ_n$ and*

$$\left| \vec{\Delta} \left(\frac{n{q_n}^{1/p'}}{Q_n} \right) \right| \leq C_p \frac{{q_n}^{1/p'}}{Q_n}$$

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for some constants A and C_p and $n = 1, 2, \ldots$. Then for each $N \geq 1$,

$$\sum_{n=1}^{N} q_n \left[\sum_{k=n}^{N} \frac{q_k a_k}{Q_k} \right]^p \leq k(p) \sum_{n=1}^{N} \left[\sum_{k=n}^{N} \frac{{q_k}^{1/p} a_k}{k} \right]^p \tag{1.4}$$

where $k(p) \leq [A + pC_p]^p$.

For example, the assumptions in Theorems B and C are met by $q_n = \frac{1}{n}$, $Q_n \approx \log n$. In this paper (Theorems 1 and 2 below) we obtain generalizations of Theorems B and C by viewing the right side of the stated inequalities to be special cases of the weighted means $\overline{t_n} = \frac{1}{P_n} \sum_{k=1}^{n} p_k {q_k}^{1/p} a_k$ and $\overline{\sigma_n} = \sum_{k=1}^{n} \frac{p_k {q_k}^{1/p} a_k}{P_k}$ where $p_k = 1$, $P_n = n$.

As another type of generalization of Theorem A, we consider the non-negative convex function $H(u)$ defined on $[0, \infty)$. In the special case $H(u) = u$, (1.1) could be expressed as $\sum_{n=1}^{\infty} q_n \left(H \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k a_k \right) \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} q_n (H(a_n))^p$.

In Theorem 3 below, we extend this result to arbitrary convex $H(u)$ and employ a weighted mean. An integral inequality with similar spirit has recently been obtained by Packpatte [P].

2. Statement of Results

In the following $K(p)$ denotes a positive constant (which may be different at different occurrences) depending on p alone, where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1 *Assume $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are non-negative sequences for $n = 1, 2, \ldots$. Let $P_n = p_1 + \ldots + p_n$ and $Q_n = q_1 + \ldots + q_n$. Denote $\vec{\Delta}\, u_k = u_k - u_{k+1}$ and $\overline{t_{n,p}} = \overline{t_n} = \frac{1}{P_n} \sum_{k=1}^{n} p_k {q_k}^{1/p} a_k$. Assume*

$$\frac{P_k}{p_k} \leq A \frac{Q_k}{q_k} \tag{2.1}$$

$$P_k \left| \vec{\Delta} \frac{{q_k}^{1/p'}}{p_k} \right| \leq B_p {q_k}^{1/p'}. \tag{2.2}$$

Then

$$\sum_{n=1}^{N} q_n \left[\frac{1}{Q_n} \sum_{k=1}^{n} q_k a_k \right]^p \leq K(p) \sum_{n=1}^{N} \overline{t}_n^p \tag{2.3}$$

where $K(p) \leq \left(A + \frac{p}{p-1} B_p \right)^p$.

Theorem 2 *Define* $\overline{\sigma}_{n,p} = \overline{\sigma}_n = \sum_{k=n}^{N} \frac{p_k {q_k}^{1/p} a_k}{P_k}$ *for* $n \le N$ *and* $\sigma_{N+1} = 0$. *With notation as in Theorem 1, assume (2.1) and the following:*

$$\left| \vec{\Delta} \left(\frac{{q_{k-1}}^{1/p'}}{Q_{k-1}} \cdot \frac{P_{k-1}}{p_{k-1}} \right) \right| \le C_p \frac{{q_k}^{1/p'}}{Q_k}. \tag{2.4}$$

Then

$$\sum_{n=1}^{N} q_n \left[\sum_{k=n}^{N} \frac{q_k a_k}{Q_k} \right]^p \le K(p) \sum_{n=1}^{N} \overline{\sigma_n}^p. \tag{2.5}$$

where $K(p) \le (A + pC_p)^p$.

Theorem 3 *Assume* $p, q > 1$ *and* $H(u)$ *a non-negative convex function defined on* $[0, \infty)$. *Then*

$$\sum_{n=1}^{N} p_n {P_n}^{p-q} \left[H \left[\frac{1}{P_n} \sum_{k=1}^{n} p_k a_k \right] \right]^p \le \left(\frac{pq}{p-1} \right)^p \sum_{n=1}^{N} p_n {P_n}^{p-q} (H(a_n))^p. \tag{2.6}$$

Let $\varepsilon \ge 0$ and $0 < \alpha < 1$. Then conditions (2.1), (2.2), and (2.4) hold for $q_n = \frac{1}{n(\log n)^\varepsilon}$ and $p_n = \frac{1}{n^\alpha}$. In this case $Q_n \approx \frac{1}{(\log n)^{\varepsilon - 1}}$ for $\varepsilon \ne 1$ and $Q_n \approx \log(\log n)$ for $\varepsilon = 1$, while $P_n \approx \frac{1}{n^{\alpha - 1}}$. Corollary 1 illustrates Theorem 1 in the case $\varepsilon = 0$.

Corollary 1 *If* $q_n = \frac{1}{n}$ *and* $p_n = \frac{1}{n^\alpha}$ *for* $\alpha > 1$ *then for* $a_k \ge 0$

$$\sum_{n=1}^{N} \frac{1}{n} \left[\frac{1}{\log(n+1)} \sum_{k=1}^{n} \frac{a_k}{k} \right]^p \le k(p) \sum_{n=1}^{N} \left[\frac{1}{n^{\alpha-1}} \sum_{k=1}^{n} \frac{a_k}{k^{\alpha + 1/p}} \right]^p.$$

A similar corollary can be mentioned for Theorem 2.

We remark that in the case $p_n = 1$, $P_n = \frac{1}{n}$ we obtain Theorems 1 and 2 of [MRV], although the condition in Theorem 2 of that paper differs slightly from (2.4). Furthermore, the conditions (2.1), (2.2), and (2.4) are satisfied by $q_n = \frac{1}{n^\beta}$, $p_n = \frac{1}{n^\alpha}$ with $0 \le \alpha$, $\beta < 1$. However, the resulting inequalities may be obtained directly from [MRV] by choosing $q_n = \frac{1}{n^\beta}$.

In Theorem 3, if $H(u) = e^{u/p}$, $p_n = 1$ and $p = q$, we obtain the following:

Corollary 2 $\sum_{n=1}^{N} \exp\left(\frac{1}{n} \sum_{k=1}^{n} a_k\right) \le \left(\frac{p^2}{p-1}\right)^p \sum_{n=1}^{N} e^{a_n}$.

In particular, if $a_k = \log b_k$, $b_k > 0$, then

$$\sum_{n=1}^{N} \left(\prod_{k=1}^{n} b_k \right)^{1/n} \le \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{N} b_n.$$

3. Proofs

Proof of Theorem 1: Write $T_n \equiv P_n\overline{t_n} = \sum_{k=1}^n p_k q_k{}^{1/p} a_k$ with $T_0 = 0$.

$$\overleftarrow{\Delta}\, T_k := T_k - T_{k-1} = p_k q_k{}^{1/p} a_k, \; q_k a_k = \frac{q_k{}^{1/p'}}{p_k} \overleftarrow{\Delta}\, T_k \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

Hence for $\overrightarrow{\Delta}\, C_k := C_k - C_{k+1}$

$$\begin{aligned} 0 \quad &\leq \quad l_n := \frac{1}{Q_n}\sum_{k=1}^n q_k a_k = \frac{1}{Q_n}\sum_{k=1}^n \frac{q_k{}^{1/p'}}{p_k} \overleftarrow{\Delta}\, T_k \\ &= \quad \frac{1}{Q_n}\left(\sum_{k=1}^{n-1} T_k \overrightarrow{\Delta}\, \frac{q_k{}^{1/p'}}{p_k} + \frac{q_n{}^{1/p'}}{p_n} T_n\right). \end{aligned}$$

Using (2.1) and (2.2) we now have

$$l_n \leq \frac{1}{Q_n}\left[\sum_{k=1}^{n-1} B_p q_k{}^{1/p'}\overline{t_k}\right] + A q_n{}^{-1/p}\overline{t_n} := l'_n + l''_n$$

and hence by Minkowski's inequality

$$\begin{aligned} \left(\sum_{n=1}^N q_n l_n{}^p\right)^{1/p} \quad &\leq \quad \left(\sum_{n=1}^N q_n (l'_n)^p\right)^{1/p} + \left(\sum_{n=1}^N q_n (l''_n)^p\right)^{1/p} \\ &\leq \quad B_p\left(\sum_{n=1}^N q_n \left[\frac{1}{Q_n}\sum_{k=1}^n q_k{}^{1/p'}\overline{t_k}\right]^p\right)^{1/p} + A\left(\sum_{n=1}^N \overline{t_n}^p\right)^{1/p}. \end{aligned}$$

Now using Copson's inequality; that is

$$\sum_{N=1}^N q_n \left(\frac{1}{Q_n}\sum_{k=1}^n q_k b_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^N q_n b_n{}^p$$

with $b_k = q_k{}^{-1/p}\overline{t_k}$, we complete the proof with $K(p) \leq A + B_p\left(\frac{p}{p-1}\right)$.

Proof of Theorem 2: Recall $\overline{\sigma_{N+1}} = 0$ and $\overline{\sigma_n} = \sum_{k=n}^N \frac{p_k q_k{}^{1/p} a_k}{P_k}$, and hence $P_k \overrightarrow{\Delta}\, \overline{\sigma_k} = P_k(\overline{\sigma_k} - \overline{\sigma_{k+1}}) = p_k q_k{}^{1/p} a_k$. Then $q_k a_k = q_k{}^{1/p'}\left(\frac{P_k}{p_k}\right)\overrightarrow{\Delta}\, \overline{\sigma_k}$ and we have

$$\sum_{k=n}^N \frac{q_k a_k}{Q_k} \quad = \quad \sum_{k=n}^N \frac{q_k{}^{1/p'}}{Q_k}\frac{P_k}{p_k}\overrightarrow{\Delta}\, \overline{\sigma_k}$$

$$= -\sum_{k=n+1}^{N}\left(\vec{\Delta}\left(\frac{q_{k-1}{}^{1/p'}}{Q_{k-1}}\frac{P_{k-1}}{p_{k-1}}\right)\right)\overline{\sigma}_k + \overline{\sigma}_n\frac{q_n{}^{1/p'}}{Q_n}\frac{P_n}{p_n}.$$

Using (2.1) and (2.5) we have

$$\sum_{k=n}^{N}\frac{q_k a_k}{Q_k} \le C_p \sum_{k=n+1}^{N}\frac{q_k{}^{1/p'}}{Q_k}\overline{\sigma}_k + A q_n{}^{-1/p}\overline{\sigma}_n.$$

We now write, using Minkowski's inequality,

$$\left(\sum_{n=1}^{N} q_n\left[\sum_{k=n}^{N}\frac{q_k a_k}{Q_k}\right]^p\right)^{1/p} \le C_p \left(\sum_{n=1}^{N} q_n\left[\sum_{k=n+1}^{N}\frac{q_k{}^{1/p'}\overline{\sigma}_k}{Q_k}\right]^p\right)^{1/p} + A\left(\sum_{n=1}^{N}\overline{\sigma_n}^p\right)^{1/p}.$$

We now apply the second Copson's inequality, namely

$$\sum_{n=1}^{N} q_n\left(\sum_{k=n}^{N}\frac{q_k}{Q_k}b_k\right)^p \le p^p\sum_{k=1}^{N} q_k b_k{}^p$$

with $b_k = q_k{}^{-1/p}\overline{\sigma}_k$ to complete the proof with $K(p) = (A + pC_p)^p$.

For the proof of Theorem 3, we will require the following lemma:

Lemma 1 ([DP]) *If* $p > 1$ *and* $z_n \ge 0$, $n = 1, 2, \ldots$, *then*

$$\left(\sum_{k=1}^{n} z_k\right)^p \le p\sum_{k=1}^{n} z_k\left(\sum_{j=1}^{k} z_j\right)^{p-1}.$$

Proof of Theorem 3: By Jensen's inequality, since $H(u)$ is convex,

$$\sum_{n=1}^{N} p_n P_n{}^{p-q}\left(H\left(\sum_{k=1}^{n}\frac{p_k}{P_n}a_k\right)\right)^p \le \sum_{n=1}^{N} p_n P_n{}^{p-q}\left(\sum_{k=1}^{n}\frac{p_k}{P_n}H(a_k)\right)^p.$$

Now apply Lemma 1 to the larger side with $z_k = p_k H(a_k)$

$$\sum_{n=1}^{N} p_n P_n{}^{-q}\left[\sum_{k=1}^{n} p_k H(a_k)\right]^p \le p\sum_{n=1}^{N} p_n P_n{}^{-q}\sum_{k=1}^{n} p_k H(a_k)\left[\sum_{r=1}^{k} p_r H(a_r)\right]^{p-1}.$$

Rearranging and denoting $Q_n = \sum_{k=1}^{n} p_k H(a_k)$, the above inequality may be written

$$\sum_{n=1}^{N} p_n P_n{}^{-q} Q_n{}^{p} \le p \sum_{k=1}^{N} p_k H(a_k) Q_k{}^{p-1} \sum_{n=k}^{N} p_n P_n{}^{-q}.$$

Observe now that

$$\sum_{n=k}^{N} \frac{p_n}{P_n{}^q} \le \frac{p_k}{P_k{}^q} + \sum_{j=k+1}^{N} \int_{P_{j-1}}^{P_j} \frac{dx}{x^q} \le P_k{}^{1-q} + \frac{P_k{}^{1-q}}{q-1} = \frac{q}{q-1} P_k{}^{1-q}$$

and therefore

$$\sum_{n=1}^{N} p_n P_n{}^{-q} Q_n{}^{p} \le \frac{pq}{q-1} \sum_{k=1}^{N} p_k H(a_k) Q_k{}^{p-1} P_k{}^{1-q}$$

$$\sum_{n=1}^{N} p_n P_n{}^{-q} Q_n{}^{p} \le \frac{pq}{q-1} \sum_{k=1}^{N} \left(\frac{p_k}{P_k{}^q}\right)^{1/p} \left(\frac{p_k}{P_k{}^q}\right)^{1/p'} P_k H(a_k) Q_k{}^{p-1}$$

$$\sum_{n=1}^{N} p_n P_n{}^{-q} Q_n{}^{p} \le \frac{pq}{q-1} \left\{\sum_{k=1}^{N} \frac{p_k}{P_k{}^q} P_k{}^p (H(a_k))^p\right\}^{1/p} \left\{\sum_{k=1}^{N} \frac{p_k}{P_k{}^q} Q_k{}^p\right\}^{1/p'}$$

by Holder's inequality. To complete the proof, divide both sides by the last factor on the right and observe that if this factor is zero, then the theorem is certainly true.

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