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Bernd Kirchheim, Institute of applied mathematics, Comenius University, 842 15 Bratislava, Czechoslovakia

Typical Approximately Continuous Functions Are Surprisingly Thick

We show that the following properties are typical ones in the space of bounded approximately continuous functions: The Hausdorff dimension of the graph of f equals two and any level set $f^{-1}(y)$ for $\inf f < y < \sup f$ has Hausdorff dimension one. This example serves as a warning to be careful when applying plausible but unproved "interpolation principles".

1. Introduction

In [2] several problems concerning the typical behavior of functions in the various Zahorski classes were posed. Among them there was the question concerning the "metrical" size of level sets; i.e. size with respect to Hausdorff measures etc. In [9] this question was answered for the first Zahorski class $b\mathcal{M}_1 = b\mathcal{D}B^1$, i.e. the class of bounded Darboux Baire one functions. In this space "typical level sets" are very small. More precisely, for any (continuous) Hausdorff measure there is a residual set of functions f in $b\mathcal{D}B^1$ having all level sets $f^{-1}(y), y \in \mathbb{R}$, of measure zero. (See Theorem 4 in [9].) The same is true also for typical continuous functions, as shown e.g. in [1]. (Also see Corollary 1 in [9] for a different approach. Based on these two facts it was conjectured in [9] that this behavior is typical also in the intermediate space of all bounded approximately continuous functions. It is the goal of the present paper to disprove this conjecture. (See Corollary 8 below.) Simultaneously we also show that the graphs of typical approximately continuous functions are much bigger (in this metrical sense) than those of both typical $b\mathcal{D}B^1$ and typical continuous functions.

First we have to agree on some notation. By $b\mathcal{A}([0,1]), b\mathcal{D}B^1([0,1]), \mathcal{C}([0,1])$ or $b\mathcal{A}, b\mathcal{D}B^1, \mathcal{C}$ we denote the space of bounded approximately continuous, bounded Darboux Baire one, and continuous functions, respectively, all defined on $[0,1]$. These spaces are equipped with the supremum norm $\| \ \|_\infty$. For any $d > 0$

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we define the d-dimensional Hausdorff measure $\mathcal{H}^d$ as follows. If $A \subset \mathbb{R}^n$, $n \geq 1$, and $\varepsilon > 0$, then

$$\mathcal{H}^d_\varepsilon(A) = \inf\left\{\sum_{k=1}^{\infty}(diam\, A_k)^d \; ; \; \bigcup_{k=1}^{\infty} A_k \supset A \text{ and } diam\, A_k < \varepsilon \text{ for } k \geq 1\right\} .$$

Clearly, $\mathcal{H}^d_\varepsilon \geq \mathcal{H}^d_\eta$ if $0 < \varepsilon < \eta$ and hence, we can define

$$\mathcal{H}^d(A) = \lim_{\varepsilon \searrow 0} \mathcal{H}^d_\varepsilon(A).$$

Then $\mathcal{H}^d$ is a metric outer measure. Moreover, it can be easily shown that for any $A \subset \mathbb{R}^n$ there is a unique number $D \in [0, n]$ such that $\mathcal{H}^d(A) = \infty$ if $0 < d < D$ and that $\mathcal{H}^d(A) = 0$ for $d > D$. This number is said to be the Hausdorff dimension of A and denoted by $\dim_{\mathcal{H}} A$. The larger class of Hausdorff measures mentioned above in connection with the level sets of $b\mathcal{D}B^1$ or continuous functions is obtained using more general functions $\phi(diam\, A)$ instead of powers only.

The (outer) Lebesgue measure on the real line, i.e. $\mathcal{H}^1$, is denoted by $|\;|$. The upper density of the set $M \subset \mathbb{R}$ at $x \in \mathbb{R}$ is defined by $\overline{D}(M, x) = \limsup_{r \searrow 0} |(x - r, x + r) \cap M|/2r$. A very useful tool for the estimation of Hausdorff dimension is net-measure defined by means of dyadic intervals (see [5]). For $k \in \mathbb{Z}$ let $\mathcal{B}_k$ be the system of all right-open dyadic intervals of length 2^{-k}, i.e. $\mathcal{B}_k = \{[i \cdot 2^{-k}, (i+1) \cdot 2^{-k}) \; ; \; i \in \mathbb{Z}\}$. Further, for any interval $I \subset \mathbb{R}$ we put $\mathcal{B}_k(I) = \{J \in \mathcal{B}_k \; ; \; int\, I \cap J \neq \emptyset\}$. And finally, in the long formulae to follow it will often be convenient to use the expression $\exp_2(x)$ instead of 2^x.

Our procedure will be as follows. In the first step we construct sets for which the (big) dimension of their boundaries in the density topology is stable under certain modifications. This construction is the crucial step of the paper. The existence of such sets is the main tool in the second step where it is shown that for "sufficiently many" functions all level sets contain density boundaries of such sets. Finally, in the last part these results are used to obtain dimension estimates for graphs of typical functions.

2. The Construction of the Set G

In this section the following theorem is proved.

Theorem 1 *Let $M \subset [0, 1]$ be a Lebesgue measurable set with $|M| > 0$. Then there is a compact set $C \subset M$ and an open set $G \subset (0, 1)$ such that $\overline{D}(C, x) > 0$ for all $x \in C$, $|C \setminus G| > 0$ and for any open $U \subset (0, 1)$ satisfying $C \cap G \subset U$ and*

$|C \setminus U| > 0$ *we have*

$$\dim_{\mathcal{H}} \{x \notin U \ ; \ \overline{D}(U,x) \geq \frac{1}{8}\} = 1.$$

For the proof of this theorem we use the following "uniformization argument" appearing e.g. in [10] and [3].

Lemma 2 *For any $M \subset [0,1]$ measurable with $|M| > 0$ there is a compact set $C \subset M \cap (0,1)$ of positive measure and a sequence $\{n_k\}_{k=1}^{\infty}$ such that for any $k \geq 1$*

$$n_k \geq k^3 \tag{1}$$

$$n_{k+1} - n_k \geq 3k^2 + 3k + 1 \tag{2}$$

$$\textit{for each } I \in \mathcal{B}_{n_k} \ \frac{|I \setminus C|}{|I|} \leq 2^{-k-4} \ \textit{ or } I \cap C = \emptyset. \tag{3}$$

Proof. Of course, it suffices to find some sequence $\{m_k\}_{k=1}^{\infty} \nearrow \infty$ and compact subsets $C_1 \supset C_2 \supset \ldots$ of M such that for any $k \geq l \geq 1$ both $|C_k| > |M|/2$ and for $I \in \mathcal{B}_{m_l}$ either $I \cap C_k = \emptyset$ or $|I \setminus C_k| < 2^{-l-4}2^{-m_l}$. Then one can put $C = \bigcap_{k=1}^{\infty} C_k$ and choose $\{n_k\}_{k=1}^{\infty}$ to be a sufficiently rapidly growing subsequence of $\{m_k\}_{k=1}^{\infty}$. But if for some $k \geq 1$ admissible $m_1, \ldots, m_k$ and $C_1 \supset \ldots \supset C_k$ are given, then suitable $m_{k+1} > m_k$ and $C_{k+1} \subset C_k$ are easily found using Lebesgue's density theorem. More details can be found in [10].

The set C appearing in the statement of 2 is that used in 1. Note that condition (3) implies $\overline{D}(C,x) \geq 1/2$ for any $x \in C$. Now we describe how to obtain the set G.

Choose $\{n_k\}_{k=1}^{\infty}$ and C according to Lemma 2. We define $G_0 = \emptyset$ and now assume that for some $k \geq 0$ $G_0, \ldots, G_k$ have been chosen. We put

$$G_{k+1} = G_k \cup \bigcup_{I \in \mathcal{B}'_{k+1}} (\inf I, \inf I + 2^{-n_{k+1}-(k+1)}), \tag{4}$$

where

$$\mathcal{B}'_{k+1} = \{I \in \mathcal{B}_{n_{k+1}} \ ; \ I \cap G_k = \emptyset \text{ and } I \cap C \neq \emptyset\}. \tag{5}$$

According to (2), $n_{k+1} > n_k + k$. Thus $\mathcal{B}'_{k+1} = \{I \in \mathcal{B}_{n_{k+1}} \ ; \ |I \setminus G_k| \neq 0 \text{ and } I \cap C \neq \emptyset\}$. Finally, let $G = \bigcup_{k=1}^{\infty} G_k$.

For any $I \in \mathcal{B}'_{k+1}$ we have

$$|(G_{k+1} \setminus G_k) \cap I| = 2^{-k-1} \frac{|C \cap I|}{|C \cap I|/|I|} \leq 2^{-k-1} \frac{|(C \setminus G_k) \cap I|}{1 - 2^{-k-5}}$$

by (3). Since $G_{k+1} \setminus G_k \subset \bigcup \mathcal{B}'_{k+1}$, we obtain $|G_{k+1} \setminus G_k| < 2^{-k}|C \setminus G_k|$. Consequently,

$$\prod_{k=1}^{\infty} \frac{|C \setminus G_{k+1}|}{|C \setminus G_k|} \geq \prod_{k=1}^{\infty} \left(1 - \frac{|G_{k+1} \setminus G_k|}{|C \setminus G_k|}\right) \geq \prod_{k=1}^{\infty}(1 - 2^{-k}) > 0$$

and $|C \setminus G| > 0$ as required.

It remains to prove that for any open set $U \subset [0,1]$ with $G \cap C \subset U$ and $|C \setminus U| > 0$ the "density boundary" of U does have dimension one.

Lemma 3 *Let C, G and $\{n_k\}_{k=1}^{\infty}$ be as constructed above. Further, assume that $U \subset [0,1]$ is open, $U \supset C \cap G$ and $2^{-K} \leq |F|$ where $F = C \setminus U$ and $K \geq 3$. Define $p_0 = 1$ and*

$$p_{k+1} = \exp_2(n_{K+k+1} - n_{K+k} - 2(K+k) - 1) \text{ for } k \geq 0.$$

Then there is a system $\mathcal{S}$ of closed dyadic intervals

$$\mathcal{J}_{l_0,\ldots,l_k} \text{ with } 1 \leq l_i \leq p_i \text{ for } 0 \leq i \leq k \tag{6}$$

such that

$$\begin{aligned} \mathcal{J}_{l_0,\ldots,l_k} &\supset \mathcal{J}_{l_0,\ldots,l_k,l_{k+1}} , & (7)\\ \mathcal{J}_{l_0,\ldots,l_k,l} &\cap \mathcal{J}_{l_0,\ldots,l_k,l'} = \emptyset \text{ for } l \neq l' , & (8)\\ |\mathcal{J}_{l_0,\ldots,l_k}| &= \exp_2(-n_{K+k} - (K+k)) , & (9)\\ |\mathcal{J}_{l_0,\ldots,l_k} \cap F| &\geq \exp_2(-K-k)|\mathcal{J}_{l_0,\ldots,l_k}| , & (10) \end{aligned}$$

and

$$\begin{aligned} &\text{for all } x \in \mathcal{J}_{l_0,\ldots,l_k} \text{ there is a } d \in (0, 2|\mathcal{J}_{l_0,\ldots,l_k}|) \\ &\text{with } |(x-d, x+d) \cap U| \geq \tfrac{1}{8}2d . \end{aligned} \tag{11}$$

(In (7)-(11) only $l_0, \ldots, l_k, l_{k+1}$ according to (6) are considered.)

Proof. We construct $\mathcal{S}$ inductively. Because $2^{-K} \leq |F|$ there are intervals in $\mathcal{B}_{n_K+K}([0,1])$ satisfying (10) for $k = 0$. Denote by $\mathcal{J}_1$ the left most of them. Then the definition of $\mathcal{B}'_K$ and of G_K ensures that $\mathcal{J}_1$ also fulfills (11) for $k = 0$. (The argument needed will be presented in detail below.) Now assume that for some $k \geq 0$ all $\mathcal{J}_{l_0,\ldots,l_k}$ for $1 \leq l_i \leq p_i$ and $0 \leq i \leq k$ have been selected. Fix an arbitrary $\mathcal{J} = \mathcal{J}_{l_0,\ldots,l_k}$. Denote by N the number of intervals $I \in \mathcal{B}_{n_{K+k+1}}(\mathcal{J})$ such that

$$|I \cap F|/|I| \geq \frac{1}{2}|\mathcal{J} \cap F|/|\mathcal{J}| \; (\geq 2^{-(K+k+1)}) . \tag{12}$$

We infer from the evident inequality

$$\frac{|\mathcal{J}\cap F|}{|\mathcal{J}|} < \frac{\exp_2(-n_{K+k+1})}{|\mathcal{J}|}\left[N\cdot 1 + \left(\frac{|\mathcal{J}|}{\exp_2(-n_{K+k+1})} - N\right)\cdot\frac{1}{2}\frac{|\mathcal{J}\cap F|}{|\mathcal{J}|}\right]$$

that

$$\frac{1}{2}\frac{|\mathcal{J}\cap F|}{|\mathcal{J}|} < \frac{\exp_2(-n_{K+k+1})}{|\mathcal{J}|}\cdot N\left(1-\frac{1}{2}\frac{|\mathcal{J}\cap F|}{|\mathcal{J}|}\right)$$

and consequently

$$\begin{aligned} N &> \frac{|\mathcal{J}|}{\exp_2(-n_{K+k+1})}\frac{|\mathcal{J}\cap F|}{2|\mathcal{J}|} \geq \exp_2(-n_{K+k} - (K+k) + n_{K+k+1})2^{-K-k-1} \\ &= \exp_2(n_{K+k+1} - n_{K+k} - 2(K+k) - 1) = p_{k+1} \,. \end{aligned}$$

But for any $I \in \mathcal{B}_{n_{K+k+1}}(\mathcal{J})$ satisfying (12) there are $\mathcal{J}' \in \mathcal{B}_{n_{K+k+1}+(K+k+1)}(I)$ with $|F\cap\mathcal{J}'| \geq 2^{-(K+k+1)}|\mathcal{J}'|$. Denote by $\mathcal{J}_I$ the closure of the left most of them. Since $f\cap G = \emptyset$, $I \in \mathcal{B}'_{K+k+1}$. Thus the construction of G_{K+k+1} and of G ensures that $\inf I < \inf\mathcal{J}_I < \sup\mathcal{J}_I \leq \sup I$. Therefore, the intervals $\mathcal{J}_I$ are mutually disjoint. Next, let $x \in \mathcal{J}_I$ and put $d = (x - \inf\mathcal{J}_I) + |\mathcal{J}_I| \in (0, 2|\mathcal{J}_I|)$. Then $I_x = [x-d, \inf\mathcal{J}_I) \in \mathcal{B}_{n_{K+k+1}+K+k+1}(I)$ and therefore $|I_x\cap F| < |\mathcal{J}_I|\exp_2(-K-k-1)$. Since $I\cap C \neq \emptyset$, we infer from (3) that $|I_x\setminus C| \leq |I\setminus C| \leq 2^{-K-k-5}|I| = \frac{1}{16}|\mathcal{J}_I|$. Hence $|I_x\setminus(C\setminus F)| < |\mathcal{J}_I|/8$ and we conclude that $|[x-d, x+d]\cap U| \geq |I_x\cap(C\setminus F)| > \frac{7}{8}|\mathcal{J}_I| \geq \frac{7}{32}2d$. Summarizing we see that it suffices to pick any p_{k+1} of the N intervals $\mathcal{J}_I$ to form the collection $\mathcal{J}_{l_0,\ldots,l_k,l}$, for $1 \leq l \leq p_{k+1}$. In this way we construct the entire family $\{\mathcal{J}_{l_0,\ldots,l_k,l_{k+1}}\}$, for $1 \leq l_i \leq p_i$, $1 \leq i \leq k+1$ by induction.

We set

$$S = F \cap \bigcap_{k=0}^{\infty} \bigcup_{l_0,\ldots,l_k} \mathcal{J}_{l_0,\ldots,l_k}$$

and according to (11) we have $S \subset \{x \notin U\ ;\ \overline{D}(U,x) \geq \frac{1}{8}\}$. We now finish the proof of 1 giving the required estimate for S.

First, note that $S = \bigcap_{k=0}^{\infty}\bigcup_{l_0,\ldots,l_k}\mathcal{J}_{l_0,\ldots,l_k}$. Indeed, since F is closed, no $x \notin F$ can be contained in arbitrary small intervals fulfilling (10).

Next, assume $\mathcal{H}^d(S) = 0$ for some positive $d < 1$. For $k \geq 1$ define

$$P_k = \prod_{i=0}^{k} p_i = \exp_2(n_{K+k} - n_K - 2Kk - k^2), \text{ and}$$

$$h(k) = P_k\,[\exp_2(-n_{K+k} - K - k)]^d \exp_2(-2(K+k+2)d)\,. \tag{13}$$

The assumptions (1) and (2) made on $\{n_k\}_{k=1}^{\infty}$ ensure that there is a $K_1 \geq K$ satisfying

$$h(k) > 1 \text{ and } (1-d)(n_{k+1}-n_k) > 2(K+k+1)+3d \text{ for any } k \geq K_1 - K \ . \quad (14)$$

The definition of $\mathcal{H}^d$ implies the existence of a sequence $\{U_j\}_{j=1}^{\infty}$ of open intervals of length less than $\exp_2(-n_{K_1} - K_1)$ such that $S \subset \bigcup_{j=1}^{\infty} U_j$ and $\sum_{j=1}^{\infty} |U_j|^d < \frac{1}{3}$. Moreover, since S is compact, there are numbers N and $K' \geq K_1$ with

$$\bigcup_{l_0,\ldots,l_{K'-K}} \mathcal{J}_{l_0,\ldots,l_{K'-K}} \subset \bigcup_{j=1}^{N} U_j \ .$$

Furthermore, because each U_j is contained in the union of three right-open dyadic intervals of equal length not longer than $|U_j|$ and since any systems of such dyadic intervals contains a mutually disjoint one with the same union, we may assume that there is a $K_2 > K'$ and an $M \in [N, 3N]$ such that

$$\left.\begin{array}{l} T = \bigcup_{l_0,\ldots,l_{K_2-K}} \mathcal{J}_{l_0,\ldots,l_{K_2-K}} \subset \bigcup_{i=1}^{M} I_i \text{ and } \sum_{i=1}^{N} |I_i|^d < 1 \\ \text{where the } \{I_i\}_{i=1}^{M} \subset \bigcup_{j=n_{K_1}+K_1}^{n_{K_2}+K_2} \mathcal{B}_j \text{ are mutually disjoint.} \end{array}\right\} \quad (15)$$

In order to obtain a contradiction it now suffices to prove the following

<u>*Claim.*</u> For all $k \geq K_1 - K$ and for any $\mathcal{J} = \mathcal{J}_{l_0,\ldots,l_k} \in \mathcal{S}$ we have

$$\sum_{I_i \cap int\, \mathcal{J} \neq \emptyset} |I_i|^d \geq \exp_2(-2d(K+k+2))|\mathcal{J}|^d \ . \quad (16)$$

Indeed, since $I_i \cap int\, \mathcal{J}_{l_0,\ldots,l_{K_1-K}} \neq \emptyset$ iff $I_i \subset \mathcal{J}_{l_0,\ldots,l_{K_1-K}}$, for $k = K_1 - K$ the inequality (16) contradicts (15), (13) and the first statement of (14).

Of course, (16) holds for any $k > K_2 - K$ since any interval in $\mathcal{S}$ of such small size is contained in some single interval I_i. Hence, we may restrict our attention to the case of an interval $\mathcal{J} = \mathcal{J}_{l_0,\ldots,l_k}$, $k \geq K_1 - K$ such that (16) is false for $\mathcal{J}$ but is true for any subinterval $\mathcal{J}' \in \mathcal{S}$ of $\mathcal{J}$. We set

$$z = card\{l \leq p_{k+1} \ ; \ \mathcal{J}_{l_0,\ldots,l_k,l} \subset cl\, I_i \text{ for some } i \leq M\}.$$

Since (16) does not hold for $\mathcal{J}$ and since $\sqrt[d]{a} + \sqrt[d]{b} \leq \sqrt[d]{a+b}$ for any $a, b \geq 0$, we infer that

$$\begin{aligned} |\mathcal{J}| \exp_2(-2(K+k+2)) &\geq \sum_{I_i \cap int\, \mathcal{J} \neq \emptyset} |I_i| \geq z \cdot |\mathcal{J}_{l_0,\ldots,l_k,1}| \\ &= z \cdot |\mathcal{J}| \exp_2(-n_{K+k+1} + n_{K+k} - 1) \ . \end{aligned}$$

Consequently, $z \leq \exp_2(n_{K+k+1}-n_{K+k}+1-2(K+k+2)) < p_{k+1}/2$. Next, let $\mathcal{M}$ be the family of all intervals $\mathcal{J}_{l_0,\dots,l_k,l}$, with $1 \leq l \leq p_{k+1}$, which are not contained in the closure of any single interval I_i. Then $card\,\mathcal{M} = p_{k+1} - z > p_{k+1}/2$ and for any $\mathcal{J}' \in \mathcal{M}$ we have $I_i \cap int\,\mathcal{J}' \neq \emptyset$ iff $I_i \subset \mathcal{J}'$. Therefore, applying (16) to each $\mathcal{J}' \in \mathcal{M}$ we get

$$\begin{aligned}
\sum_{I_i \cap \mathcal{J} \neq \emptyset} |I_i|^d &\geq \sum_{\mathcal{J}' \in \mathcal{M}} \sum_{I_i \subset \mathcal{J}'} |I_i|^d \\
&\geq card\,\mathcal{M} \cdot \exp_2(-2d(K+k+3))(\exp_2(n_{K+k} - n_{K+k+1} - 1)|\mathcal{J}|)^d \\
&\geq |\mathcal{J}|^d \exp_2(n_{K+k+1} - n_{K+k} - 2(K+k+1) - 2d(K+k+3) \\
&\qquad\qquad -d(n_{K+k+1} - n_{K+k} + 1)) \\
&= |\mathcal{J}|^d \exp_2((1-d)(n_{K+k+1} - n_{K+k}) - 2(K+k+1) - 3d \\
&\qquad\qquad -2d(K+k+2)) \\
&\geq |\mathcal{J}|^d \exp_2(-2d(K+k+2)), \text{ by the second part of (14).}
\end{aligned}$$

Hence $\mathcal{J}$ satisfies (16). This contradiction proves the claim and shows that $\dim_{\mathcal{H}} S = 1$.

3. Main Theorem

We now turn to the second step.

Proposition 4 *Assume that the sets G and C fulfill the conclusions of Theorem 1 and that $g \in \mathcal{A}([0,1])$. If for some $t \in \mathbb{R}$ either $g(C \cap G) \subset (-\infty, t)$ and $g(C) \cap (t,\infty) \neq \emptyset$ or $g(C \cap G) \subset (t,\infty)$ and $g(C) \cap (-\infty,t) \neq \emptyset$, then* $\dim_{\mathcal{H}} g^{-1}(t) = 1$.

Proof. We study only the case $g(C \cap G) \subset (-\infty,t)$ and $C \cap g^{-1}((t,\infty)) \neq \emptyset$, the second case being similar. Denote $M = g^{-1}((-\infty,t)) \cap (0,1)$ and for $n \geq 1$ let

$$M_n = \{x \in (0,1)\ ; \quad \text{there are} \quad a,b \in (0,1) \text{ with } x \in (a,b),\ b-a < \frac{1}{n} \text{ and } |M \cap (a,b)| > \frac{b-a}{2}\}\ .$$

Since $g \in \mathcal{A}$, M is density open and therefore, for each $n \geq 1$ we have M_n is an open superset of M. Lebesgue's density theorem implies that $|M_n| \searrow |M|$ and hence, also $|M_n \cap C| \to |M \cap C|$ for $n \to \infty$. Next, there is an $x \in C$ with $g(x) > t$. Since $\overline{D}(C,x) > 0$ and $g \in \mathcal{A}$, we infer that $|C \cap g^{-1}((t,\infty))| > 0$ and $|C \cap M| < |C|$. Consequently, there is an $N \geq 1$ with $|M_N \cap C| < |C|$.

This shows that for $U = M_N \supset M \supset C \cap G$ the inequality $|C \setminus U| > 0$ holds. According to Theorem 1 the set $B = \{x \notin U \; ; \; \overline{D}(U, x) \geq \frac{1}{8}\}$ has dimension one. Now we finish the proof showing that the set B' of all $x \in B \cap (0,1)$ which are not endpoints of any component interval of U fulfills $g(B') = \{t\}$.

It is a simple observation that any finite system of intervals contains a subsystem with the same union such that no point belongs to three different intervals from this subsystem. This fact and the definition of M_N imply that

$$\text{for any component } I \text{ of } U \quad |M \cap I| \geq \frac{1}{4}|I|. \tag{17}$$

Now, for any $x \in B'$ there are sequences $r_n^+, r_n^- \searrow 0$ such that $x + r_n^+ \notin U$ and $x - r_n^- \notin U$ for $n \geq 1$. Next, given any $\varepsilon > 0$ there are $n \geq 1$ and $r \in (0, \varepsilon)$ with $r_n^+, r_n^- \in (r, \varepsilon)$ and $|[x - r, x + r] \cap U| > \frac{1}{9}2r$. Then $|U \cap [x, x + r]| > \frac{1}{9}r$ or $|U \cap [x - r, x]| > \frac{1}{9}r$. Suppose the former. If $x + r \notin U$, then let $y = x + r$. Otherwise let $y = \sup\{y' \; ; \; [x + r, y') \subset U\}$. In either case let $\mathcal{F}$ be the family of all components of $U \cap (x, y)$. Then $y - x \leq r_n^+ < \varepsilon$ and

$$\begin{aligned}
\frac{1}{9} < \frac{1}{r}|U \cap [x, x + r]| &\leq \frac{1}{y - x}|U \cap (x, y)| = \frac{1}{y - x}\sum_{I \in \mathcal{F}} |I| \\
&\leq \frac{4}{y - x}\sum_{I \in \mathcal{F}} |M \cap I|, \text{ according to (17),} \\
&\leq \frac{4}{y - x}|(x, y) \cap M| \,.
\end{aligned}$$

Similarly, in the latter case we find $r' \in (0, \varepsilon)$ with $\frac{1}{36} < |(x - r', x) \cap M|/r'$. Consequently, $\overline{D}(M, x) \geq 1/72$ for any $x \in B'$. Since g is approximately continuous at x and $g(M) \subset (-\infty, t)$, we conclude $g(x) \leq t$. On the other hand, $x \notin U$ implies $x \notin M$ and $g(x) \geq t$. Consequently, $g(x) = t$ and $g(B') = \{t\}$ as required.

Theorem 5 *Denote by $\mathcal{G}$ the interior of the set of all $f \in b\mathcal{A}([0,1])$ fulfilling* $\dim_{\mathcal{H}} f^{-1}(y) = 1$ *whenever* $\inf f < y < \sup f$. *Then $\mathcal{G}$ is dense in $b\mathcal{A}([0,1])$.*

Proof. Let $f \in b\mathcal{A}$ and denote $a = \inf f$ and $b = \sup f$. Further, let $N > 4$ be given and put $\varepsilon = (b - a)/N$.

For any $i = 1, \ldots, N$ we choose $C_i \subset f^{-1}((a+(i-1)\varepsilon, a+i\varepsilon))$ and $G_i \subset (0,1)$ fulfilling the conclusion of 1. Hence, we can find two different points x_i^+, x_i^- of density of $C_i \setminus G_i$. According to the complete regularity of the density topology, see e.g. Theorem 6.9(a) in [8], we can select a function $h_i \in b\mathcal{A}$ mapping

$[0,1]$ onto $[-1,1]$ such that $h_i(x_i^+) = 1$, $h_i(x_i^-) = -1$ and $h_i(x) = 0$ for $x \in [0,1] \setminus (C_i \setminus G_i)$. We put

$$h = 8\varepsilon h_1 + 4\varepsilon \sum_{i=2}^{N-1} h_i + 8\varepsilon h_N \text{ and } \tilde{f} = f + h .$$

Clearly $|\tilde{f} - f| < 8\varepsilon$. So it suffices to show that $\tilde{f} \in \mathcal{G}$. To this end let $g \in b\mathcal{A}$ with $\|\tilde{f} - g\|_\infty < \varepsilon$ and set $a' = \inf g$ and $b' = \sup g$. We have to distinguish several cases.

(i) If $a' < y < a - \varepsilon$, then $g(C_1 \cap G_1) \subset (y, \infty)$. As one easily verifies $g(x_1^-) < a - 6\varepsilon$ and for $x \in [0,1] \setminus (C_1 \setminus G_1)$, $g(x) > a - 6\varepsilon$. Thus $a' = \inf g(C_1 \setminus G_1)$. Consequently $g(C_1) \cap (-\infty, y)$ is nonvoid and $\dim_{\mathcal{H}} g^{-1}(y) = 1$ by 4.

(ii) If $a + (i-1)\varepsilon \le y \le a + i\varepsilon$ for $i = 0, \dots, N-2$, then $g(C_{i+2} \cap G_{i+2}) \subset (y, \infty)$ and $x_{i+2}^- \in C_{i+2} \cap g^{-1}((-\infty, y))$. From 4 we infer $\dim_{\mathcal{H}} g^{-1}(y) = 1$.

(iii) If $a + i\varepsilon \le y \le a + (i+1)\varepsilon$ for $i = 2, \dots, N$, then $g(C_{i-1} \cap G_{i-1}) \subset (-\infty, y)$ and $x_{i-1}^+ \in C_{i-1} \cap g^{-1}((y, \infty))$. Hence,4 implies $\dim_{\mathcal{H}} g^{-1}(y) = 1$.

(iv) If $b + \varepsilon < y < b'$, then similarly to i) we infer $g(C_N \cap G_N) \subset (-\infty, y)$, $g(C_N) \cap (y, \infty) \ne \emptyset$, and hence, $\dim_{\mathcal{H}} g^{-1}(y) = 1$.

Since for any $y \in (\inf g, \sup g)$ at least one of the cases i),…,iv) occurs, our proof is complete.

Corollary 6 *a) For typical $f \in b\mathcal{A}([0,1])$ we have*

$$f^{-1}(y) \begin{cases} \textit{is empty} & \textit{if } y > \sup f \textit{ or } y < \inf f, \\ \textit{is a singleton} & \textit{if } y = \sup f \textit{ or } y = \inf f, \\ \textit{has dimension 1} & \textit{if } \sup f > y > \inf f . \end{cases}$$

b) The functions having all level sets either void or of dimension one form a dense subset in $b\mathcal{A}([0,1])$.

Proof.
a) According to [4] the set $\tilde{\mathcal{G}}$ of all $f \in b\mathcal{A}$ for which both $f^{-1}(\sup f)$ and $f^{-1}(\inf f)$ are singletons is residual in $b\mathcal{A}$. Obviously, any $f \in \mathcal{G} \cap \tilde{\mathcal{G}}$ ($\mathcal{G}$ from Theorem 7) satisfies the conclusion of part a).
b) Let $f \in b\mathcal{A}$ and $\varepsilon > 0$. Select $g \in \mathcal{G}$ with $\|f - g\|_\infty < \varepsilon/2$. Set $h(x) = \min\{\sup g - \frac{\varepsilon}{2}, \max\{\inf g + \frac{\varepsilon}{2}, g(x)\}\}$. Obviously $h \in b\mathcal{A}$, $\|h - f\|_\infty < \varepsilon$ and each nonempty level set of h contains $g^{-1}(t)$ for some $t \in (\inf g, \sup g)$.

4. Consequences

We start the last part of this paper with a statement describing the dimension of the graph of a typical function in both the "large" space $b\mathcal{DB}^1$ and the "small" space $\mathcal{C}$. It was shown in [7] that the graph of a typical continuous function has dimension 1.

Lemma 7 *Let $X = b\mathcal{DB}^1([0,1])$ or $X = \mathcal{C}([0,1])$. Then for typical f in X $\dim_{\mathcal{H}} graph(f) = 1$.*

Proof. Put $M_n = \{f \in X\ ;\ \mathcal{H}^{1+n^{-1}}_{n^{-1}}(\{(x,y)\ ;\ |y - f(x)| < \delta\}) < \frac{1}{n}$ for some $\delta > 0\}$ for $n \geq 1$. Obviously, each M_n is open in X and $f \in \bigcap_{n=1}^{\infty} M_n$ implies $\mathcal{H}^{1+\varepsilon}(\mathrm{graph}(f)) = 0$ for each $\varepsilon > 0$. Since $\mathrm{proj}_x(\mathrm{graph}(f)) = [0,1]$, we conclude $\dim_{\mathcal{H}} \mathrm{graph}(f) = 1$. Hence, it remains to show that each M_n is dense in X. For $X = \mathcal{C}([0,1])$ it may be left to the reader to show (or to believe) that every function of bounded variation belongs to each M_n.

To deal with the case $X = b\mathcal{DB}^1$ we recall the following fact which is derived in [9]. (See the proof of Theorem 4 there.) For $\varepsilon > 0$ let S_ε be the family of all $f \in b\mathcal{DB}^1$ such that there is a sequence $\{I_i\}_{i=1}^{\infty}$ of intervals fulfilling

(i) $f([0,1] \setminus \bigcup_{i=1}^{\infty} I_i)$ is finite, and

(ii) $\sum_{i=1}^{\infty} |I_i|^{\varepsilon} < \varepsilon$.

Then each $S_\varepsilon, \varepsilon > 0$ is dense in $b\mathcal{DB}^1$.

Further, one easily sees that for any interval $I \subset [0,1]$ with $|I| < 1/2n$

$$\begin{aligned} \mathcal{H}^{1+n^{-1}}_{n^{-1}}([0,1] \times I) \quad , \quad \mathcal{H}^{1+n^{-1}}_{n^{-1}}(I \times [0,1]) &\leq (\sqrt{2}|I|)^{1+n^{-1}} \left(\frac{1}{|I|} + 1\right) \\ &\leq 2|I|^{1+n^{-1}}(2/|I|) \leq 4\sqrt[n]{|I|}\,. \end{aligned}$$

From these estimates and the σ-subadditivity of $\mathcal{H}^{1+n^{-1}}_{n^{-1}}$ we immediate conclude that $S_{1/5n} \subset M_n$ for each $n \geq 1$. Consequently, every M_n is dense in $b\mathcal{DB}^1([0,1])$ and our proof is complete.

The following result demonstrating the quite opposite situation in the intermediate space $b\mathcal{A}([0,1])$ is an easy consequence of Theorem 7.

Corollary 8 *The equality $\dim_{\mathcal{H}} graph(f) = 2$ holds for any function f in the open dense subset $\mathcal{G}$ of $b\mathcal{A}([0,1])$ occurring in 5.*

The definition of $\mathcal{G}$ ensures that $\inf f < \sup f$ for all $f \in \mathcal{G}$. Hence, our statement is an immediate consequence of the following "Fubini-type" dimension estimate which is a special version of Theorem 2.10.25 in [6] using the $(1-\varepsilon)$-th power of the Euclidean metric on $\mathbb{R}$ and $\mathbb{R}^2$:

For any $\varepsilon \in (0,1)$

$$\int^*_{(\inf f, \sup f)} \mathcal{H}^{1-\varepsilon}(f^{-1}(t))\, d\mathcal{H}^{1-\varepsilon}(t) \leq \mathcal{H}^{2-2\varepsilon}(\mathrm{graph}(f)) ,$$

where $\int^*$ denotes the upper integral.

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