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Symmetric Behavior in Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is symmetric at a point p iff

$$\lim_{h \to 0} (f(p+h) + f(p-h) - 2f(p)) = 0.$$

Refer to [2] for the definitions of smoothness, symmetric continuity and symmetric derivative.

Throughout our discussion $SM(f)$, $S(f)$, $SD(f)$, and $SC(f)$ will denote the set of points where a function f is smooth, symmetric, symmetrically differentiable, and symmetrically continuous, respectively.

In 1964, Stein and Zygmund in [6] showed that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and f is symmetric on a Lebesgue measurable set M, then f is continuous a.e. on M. In 1982, Evans and Larson in [4] showed the category analogue of the Stein-Zygmund theorem. Theorems 1 and 3 in [1] show that in the above theorems the additional hypothesis of M being measurable (or having the Baire property in the wide sense for the Evans-Larson theorem) is not necessary.

In 1964, Neugebauer in [5] showed that if $f : \mathbb{R} \to \mathbb{R}$ is symmetric (on $\mathbb{R}$) and Lebesgue measurable, then f has to be of Baire class 1. In 1982, Evans and Larson in [4] showed this result for a function which has the Baire property. It follows from the Evans-Larson theorem and the Neugebauer theorem that a symmetric function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable iff it has the Baire property in the wide sense. The following example of [1] shows that this is not the case if the function is not symmetric on the entire line.

Example 1 *Under the assumption of the continuum hypothesis, there exists a universally measurable function $f : \mathbb{R} \to \mathbb{R}$ which is symmetric on a second category set, yet $f|M$ is not continuous for every second category set M which has the Baire property in the wide sense. Note that for such f, $S(f) \setminus C(f)$ is second category.*

Example 9 of [1] is the category analogue of the above example.

Marcus submitted the following problem at the Summer Symposium on Real Analysis at Smolenice, Czechoslovakia, August, 1991: Given an arbitrary real function characterize its points of symmetric continuity. In [2] we show that there is no such characterization of topological nature, i.e.

Theorem 1 *Let M be a zero dimensional subset of $\mathbb{R}$. Then, there is a function $f : \mathbb{R} \to \mathbb{R}$ such that M is topologically equivalent to $SD(|f|)$, $SC(|f|)$, $SM(f)$ and $S(f)$.*

However, if we put some restriction on f then more can be said about $S(f)$ and $SC(f)$.

Theorem 2 *Suppose $f : \mathbb{R} \to \mathbb{R}$, then we have the following:*

(i) If f is of Baire class one, then S(f) and SC(f) are $G_{\delta\sigma\delta}$.

(ii) If f is Borel measurable, then S(f) and SC(f) are coanalytic.

The following theorem of [3] shows that the above theorem can not be improved.

Theorem 3 *Let M be a zero-dimensional coanalytic subset of $\mathbb{R}$. Then, there exists a Baire 2 function f such that each of $SD(|f|)$, $SC(|f|)$, $SM(f)$ and $S(f)$ is homeomorphic to M.*

References

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[4] M. J. Evans and L. Larson, *The continuity of symmetric and smooth functions*, Acta Math. Hung. **43** (3-4) (1984), 251-257.

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[6] E. M. Stein and A. Zygmund, *On the differentiability of functions*, Studia Math. **23** (1964), 247-283.