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On Approximate Peano Derivatives

Definition. We say that a function f defined on $\mathbb{R}$ has a k-th approximate Peano derivative at some point $x \in \mathbb{R}$ if there are numbers $f_1(x), \cdots, f_k(x)$ and a set V_x of density 1 at x so that

$$f(x+t) = f(x) + tf_1(x) + \cdots + \frac{t^k}{k!}f_k(x) + t^k\epsilon_k(x,t)$$

where $\lim_{x+t \in V_x,\, t \to 0} \epsilon_k(x,t) = 0$. The coefficient $f_k(x)$ is called the k-th approximate Peano derivative of f at x.

In [2] the authors introduced the concept of a path derivative as a unifying approach to the study of a number of generalized derivatives. Namely since many generalized derivatives like approximate and Peano derivatives possess most of the properties of ordinary derivatives, the authors in [2] were looking for a framework within which all of these derivatives could be presented.

The perspective they chose was to consider just those derivatives of a function F at a point x which can be obtained as

$$\lim_{y \in E_x,\, y \to x} \frac{F(y) - F(x)}{y - x}$$

for appropriate choices of sets E_x. One generalized derivative, then, differs from another only by the choice of the family of sets $\{E_x : x \in \mathbb{R}\}$ through which the difference quotient passes to its limit. For example, an approximately differentiable function F permits a choice of sets $\{E_x : x \in \mathbb{R}\}$ so that each E_x has density 1 at x; for a Dini derivative each set may consist only of a sequence converging to x. This framework includes any generalized derivative for which the derivative at a point is a derived number of the function at that point. Since Mařík has proved that $f_k(x)$ is a derived number of f_{k-1} at a point x, we see that this concept of path derivatives also includes k-th approximate Peano derivatives. (See [3].)

But in order to get some properties for path derivatives, like those possessed by Peano or approximate derivatives, we require that the family of sets $\{E_x : x \in \mathbb{R}\}$ satisfy various "thickness" conditions. These conditions relate to the "thickness" of each of the sets E_x and the way in which two of the sets intersect. The authors [2] proved that path derivatives with certain type of conditions imposed on the family of sets $\{E_x : x \in \mathbb{R}\}$, have many of the properties possessed by approximate and Peano derivatives. We will show that approximate Peano derivatives are path derivatives with $\{E_x : x \in \mathbb{R}\}$ satisfying some of the intersection conditions introduced by the authors mentioned above. In proving this assertion, we won't use any known results for approximate Peano derivatives. So this can be regarded as a new approach to studying approximate Peano derivatives. Namely all of the properties possessed by approximate Peano derivatives that are known will be obtained directly from the corresponding properties of path derivatives. The main tool will be a decomposition of approximate Peano derivatives which we will discuss next.

Let C be the family of all continuous functions on $\mathbb{R}$, Δ the family of all differentiable functions on $\mathbb{R}$ and Δ' the family of all derivatives on $\mathbb{R}$. If Γ is a family of functions defined on $\mathbb{R}$, then by $[\Gamma]$ we denote the family of all functions f on $\mathbb{R}$ with the following property: for each $n \in \mathbb{N}$ there exist $v_n \in \Gamma$ and a closed set A_n such that $f = v_n$ on A_n and $\cup_{n=1}^{\infty} A_n = \mathbb{R}$. In [1] (Theorem 2) it is shown that the following four conditions are equivalent:

(i) There are g, h and k in Δ such that $h', k' \in [C]$ and $f = g' + hk'$.

(ii) There is a $\varphi \in \Delta'$ and $\psi \in [C]$ such that $f = \varphi + \psi$.

(iii) The function $f \in [\Delta']$.

(iv) There is a dense open set T such that f is a derivative on T and f is a derivative on $\mathbb{R} \setminus T$ with respect to $\mathbb{R} \setminus T$.

In this paper we show that approximate Peano derivatives are in $[\Delta']$ and hence they are Baire 1 functions. This will enable us to show that a k-th approximate Peano derivatives, f_k, is a path derivative of the $(k-1)$-st one with respect to the nonporous system of paths satisfying $(k-1)$-st one with respect to the nonporous system of paths satisfying the I.I.C. condition. In particular f_k is a selective derivative of f_{k-1}.

References

[1] S. Agronsky, R. Biskner, A. Bruckner and J. Mařík, *Representations of functions by derivatives*, Trans. Amer. Math. Soc. **263** (1981), 493-500.

[2] A. M. Bruckner, R. J. O'Malley and B. S. Thompson, *Path derivatives: A unified view of certain generalized derivatives*, Trans. Amer. Math. Soc. **283** (1984), 97-125.

[3] J. Mařík, *On generalized derivatives*, Real Analysis Exchange **3** (1977-78), 87-92.