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On the Baire Class of E–Derivatives and Extreme E–Derivatives

The notion of path derivatives was introduced in [5] to unify the study of certain generalized derivatives. Let $x \in \mathbb{R}$. A path at x is a set $E_x \subseteq \mathbb{R}$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths is a collection $E = \{E_x : x \in \mathbb{R}\}$ where each E_x is a path leading to x. A function $F : \mathbb{R} \to \mathbb{R}$ is E–differentiable at x if $F'_E(x) = \lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y)-F(x)}{y-x}$ exists and is finite. The extreme E–derivatives are defined in the usual manner. A path system E will be said to satisfythe condition stated below if there is a positive function δ such that whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$, the paths E_x and E_y intersect in the stated fashion.

(a) *internal intersection condition* (I.I.C.) if $E_x \cap E_y \cap (x, y) \neq \emptyset$,

(b) *intersection condition* (I.C.) if $E_x \cap E_y \cap [x, y] \neq \emptyset$,

(c) *external intersection condition, parameter m* (E.I.C.[m]) if $E_x \cap E_y \cap [x - m(y-x), x] \neq \emptyset$ and $E_x \cap E_y \cap [y, y + m(y-x)] \neq \emptyset$,

(d) *one sided external intersection, parameter m* (O.E.I.C. [m]) if $E_x \cap E_y \cap [x - m(y-x), x] \neq \emptyset$ or $E_x \cap E_y \cap [y, y + m(y-x)] \neq \emptyset$.

In [5] it is shown that much of the good behavior of some generalized derivatives is due to the following factors: (i) the thickness of the paths and (ii) the way that the paths E_x and E_y intersect when x and y are sufficiently close. In [1] we studied a third factor: (iii) the closeness of the paths when x and y are sufficiently close. The factors (i) and (iii) have been studied for path derivatives and extreme path derivatives in [1], [2], [5], [3], and [4], but the intersection conditions are studied only for path derivatives [5], [6]. We mention some results from [5] and [6].

Theorem [5] *Let $F : \mathbb{R} \to \mathbb{R}$ and $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths.*

1. *If E satifies the E.I.C. and F is E–differentiable, then $F'_E \in \mathcal{B}_1$.*
2. *If E satisfies the I.I.C. and F is E–differentiable, then there exists a selection S such that $F'_E = F'_S$; hence $F'_E \in \mathcal{B}_2$.*

Theorem [6] *Let $F : \mathbb{R} \to \mathbb{R}$ and $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths satisfying the E.I.C. If F is E-differentiable, then there is a system of paths E^* which satisfies both the intersection and the external intersections so that F'_{E^*} exists and $F'_E = F'_{E^*}$.*

Thus, the following question arises:

Question 1 *Suppose $F : \mathbb{R} \to \mathbb{R}$ and $E = \{E_x : x \in \mathbb{R}\}$ is a system of paths satisfying the I.C. If F is E-differentiable, is $F'_E \in \mathcal{B}_2$?*

Of course, F'_E need not be in $\mathcal{B}_1$. This is shown in [6]. We have the following results which illustrate that the intersection conditions alone are not effective enough to guarantee the Borel measurability of extreme E-derivatives.

Theorem 1 *There exists a function $F \in \mathcal{DB}_2$ and a bilateral system of paths E satisfying the one sided E.I.C. such that $\overline{F}'_E$ is not measurable.*

Theorem 2 *There exists a function $F \in \mathcal{DB}_1$ and a bilateral system of paths E which is nonporous and satisfies all sorts of intersection conditions, but $\underline{F}'_E$ is not measurable.*

References

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